On the Relation between Possibilistic Logic and Abstract Dialectical Frameworks

Jesse Heyninck[1,2], **Matthias Thimm**[3], **Gabriele Kern-Isberner**[1], **Tjitze Rienstra**[3], **Kenneth Skiba**[3]

[1]Technische Universität Dortmund, Dortmund, Germany
[2]University of Cape Town and CAIR, South-Africa
[3]University of Koblenz-Landau, Koblenz, Germany

Abstract

Abstract dialectical frameworks (in short, ADFs) are one of the most general and unifying approaches to formal argumentation. As the semantics of ADFs are based on three-valued interpretations, the question poses itself as to whether some and which monotonic three-valued logic underlies ADFs, in the sense that it allows to capture the main semantic concepts underlying ADFs. As an entry-point for such an investigation, we take the concept of *model* of an ADF, which was originally formulated on the basis of Kleene's three-valued logic. We show that an optimal concept of a model arises when instead of Kleene's three-valued logic, possibilistic logic is used. We then show that in fact, possibilistic logic is the most conservative three-valued logic that fulfils this property, and that possibilistic logic can faithfully encode all other semantical concepts for ADFs. Based on this result, we also make some observations on strong equivalence and introduce possibilistic ADFs.

1 Introduction

Formal argumentation is one of the major approaches to knowledge representation. In the seminal paper (Dung 1995), *abstract argumentation frameworks* were conceived of as directed graphs where nodes represent arguments and edges between these nodes represent attacks. So-called *argumentation semantics* determine which sets of arguments can be reasonably upheld together given such an argumentation graph. Various authors have remarked that other relations between arguments are worth consideration. For example, in (Cayrol and Lagasquie-Schiex 2005), *bipolar argumentation frameworks* are developed, where arguments can support as well as attack each other. The last decades saw a proliferation of such extensions of the original formalism of (Dung 1995), and it has often proven hard to compare the resulting different dialects of the argumentation formalisms. To cope with the resulting multiplicity, (Brewka and Woltran 2010; Brewka et al. 2013) introduced *abstract dialectical argumentation* that aims to unify these different dialects. Just like in (Dung 1995), *abstract dialectical frameworks* (in short, ADFs) are directed graphs. In contradistinction to abstract argumentation frameworks, however, in ADFs, edges between nodes do not necessarily represent attacks but can encode *any* relationship between arguments. Such a generality is achieved by associating an *acceptance condition* with each argument, which is a Boolean formula in terms of the parents of the argument that expresses the conditions under which an argument can be accepted. As such, ADFs are able to capture all the major extensions of abstract argumentation and offer a general framework for argumentation based inference.

The semantics of ADFs are based on three-valued interpretations assigning one of three truth values true (T), false (F), and undecided (U) to arguments. Even though in various papers on ADFs, Kleene's three-valued logic is mentioned (Brewka et al. 2013; Polberg, Wallner, and Woltran 2013; Linsbichler 2014), it is not so clear what the exact role of this logic is, or for that matter any other monotonic three-valued logic, in ADFs. In this paper, we make an in-depth investigation of which three-valued logics underlie abstract dialectical frameworks, i.e. which three-valued logics allow to straightforwardly encode all semantical concepts used in ADFs. The entry point of this investigation is the notion of a *model of an ADF*, which was mentioned in (Brewka et al. 2013) but barely considered afterwards. We show that, in contradistinction to a claim made by (Brewka et al. 2013), the notion of a model of an ADF as based on Kleene's three-valued logic is ill-conceived, in the sense that it does not form a generalization of the set of admissible interpretations. We then investigate on which logics a sound notion of model can be based, and show that no truth-functional three-valued logic using an involutive negation allows to formulate an adequate concept of model for an ADF. Furthermore, we show that possibilistic logic (Dubois and Prade 1998) is able to provide an adequate notion of model. In fact, this is the least informative logic to provide such a notion. Possibilistic logic can therefore be viewed as a monotonic base logic underlying ADFs. Based on this observation, we characterize strong equivalence of ADFs and we generalize the semantics of ADFs to allow for *possibility distributions* as generalized three-valued interpretations as a basic semantic unit for ADFs. We illustrate the fruitfulness of this generalization by allowing for possibilistic constraints on arguments.

Outline of this paper: We first state all the necessary preliminaries in Section 2 on propositional logic (Sec. 2.1), three-valued logics (Sec. 2.2), possibility theory (Sec.2.3) and ADFs (Sec. 2.4). In Section 3 we answer the question which logics underlie ADFs, by first recalling and generalizing the notion of model for an ADF (Sec. 3.1), then showing that possibilistic logic underlies ADFs in Section 3.2 and thereafter making a study of the relation between truth-functional three-valued logics and ADFs, starting with some general observations (Sec. 3.3) before moving to more specific results on three-valued logics using an involutive, paraconsistent and intuitionistic negation. Thereafter, we use the fact that possibilistic logic underlies ADFs to draw some consequences on strong equivalence for ADFs (Sec. 4) and generalize the semantics of ADFs to allow for possibility distributions instead of three-valued interpretations as a basic semantic unit, allowing for a generalization of ADFs we call *possibilistic ADFs* in Sec. 5. Related work is discussed in Sec. 6 and in Sec. 7 the paper is concluded.

2 Preliminaries

In this section the necessary preliminaries on propositional logic (Section 2.1), three-valued logics (Section 2.2), possibility theory (Section 2.3), and abstract dialectical argumentation (Section 2.4) are introduced.

2.1 Propositional logic

For a set At of atoms let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), and $\neg$ (*negation*). A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\mathsf{At})$ is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{T}, \mathsf{F}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. $\mathsf{At}(\phi)$ is the set of all atoms used in a formula $\phi \in \mathcal{L}(\mathsf{At})$. We simply write Ω if the set of atoms is implicitly given. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{T}$. The satisfaction relation $\models$ is extended to formulas as usual.

As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i.e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those atoms that are assigned T by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned F by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this).

For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define the set of models $[X] = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \vdash_{\mathsf{PL}} X_2$, if $[X_1] \subseteq [X_2]$.

2.2 Three-valued logics

A 3-valued interpretation for a set of atoms At is a function $v : \mathsf{At} \rightarrow \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$, which assigns to each atom in At either the value T (true, accepted), F (false, rejected), or U (unknown). The set of all three-valued interpretations for a set of atoms At is denoted by $\mathcal{V}(\mathsf{At})$. A 3-valued interpretation v can be extended to arbitrary propositional formulas over At using various *logic systems* L. Therefore, we will, given an interpretation $v \in \mathcal{V}(\mathsf{At})$, denote the truth-value assigned by a logic system L to a formula ϕ as $v^{\mathsf{L}}(\phi)$.[1] Thus, a logic system L is defined as a function assigning a truth value to every formula-interpretation-pair. The (three-valued) models of a formula $\phi \in \mathcal{L}(\mathsf{At})$ for a logic system L are defined as $\mathcal{V}^{\mathsf{L}}(\phi) = \{v \in \mathcal{V}(\mathsf{At}) \mid v^{\mathsf{L}}(\phi) = \mathsf{T}\}$.[2] A consequence relation $\vdash_{\mathsf{L}} \subseteq \wp(\mathsf{L}(\mathsf{At})) \times \mathsf{L}(\mathsf{At}))$ can then be defined as usual by setting $\Gamma \vdash_{\mathsf{L}} \phi$ iff $\mathcal{V}^{\mathsf{L}}(\phi) \supseteq \bigcap_{\gamma \in \Gamma} \mathcal{V}^{\mathsf{L}}(\gamma)$. Thus, a logic system $\mathsf{L} : \mathcal{V}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \rightarrow \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ gives rise to a consequence relation which is most commonly associated with a logic, and we shall therefore often refer to logic systems as simply *logics*.

A particular useful class of logics are truth-functional logics:

Definition 1. We say a three-valued logic L is *truth-functional* for an n-ary connective $*$, if for every $\phi_1, \ldots, \phi_n, \phi'_1, \ldots, \phi'_n \in \mathcal{L}(\mathsf{At})$, $v^{\mathsf{L}}(\phi_i) = v^{\mathsf{L}}(\phi'_i)$ for every $1 \leq i \leq n$ implies $v^{\mathsf{L}}(*(\phi_1, \ldots, \phi_n)) = v^{\mathsf{L}}(*(\phi'_1, \ldots, \phi'_n))$.

We also introduce a rather weak notion of relevance, which expresses that the truth-value of atoms not occurring in a formula ϕ should not have any impact on the truth-value assigned by L to that formula ϕ.

Definition 2. A logic L satisfies *relevance* iff for any $\phi \in \mathcal{L}(\mathsf{At})$ and $s \in \mathsf{At}$, if $s \notin \mathsf{At}(\phi)$ then for any $v_1, v_2 \in \mathcal{V}(\mathsf{At})$, $v_1(s') = v_2(s')$ for any $s' \in \mathsf{At} \setminus \{s\}$ implies $v_1^{\mathsf{L}}(\phi) = v_2^{\mathsf{L}}(\phi)$.

This notion of relevance is very similar to the property of *independence* (Kern-Isberner, Beierle, and Brewka 2020). Notice that any truth-functional logic satisfies relevance.

We assume two commonly-used orders $\leq_i$ and $\leq_{\mathsf{T}}$ over $\{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$. $\leq_i$ is obtained by making U the minimal element: $\mathsf{U} <_i \mathsf{T}$ and $\mathsf{U} <_i \mathsf{F}$ and this order is lifted pointwise as follows (given two valuations v, w over At): $v \leq_i w$ iff $v(s) \leq_i w(s)$ for every $s \in \mathsf{At}$. $\leq_{\mathsf{T}}$ is defined by $\mathsf{F} \leq_{\mathsf{T}} \mathsf{U} \leq_{\mathsf{T}} \mathsf{T}$ and can be lifted pointwise in a similar fashion.

It will sometimes prove useful to compare logics w.r.t. their *conservativeness*:

Definition 3. Given two logics L and L', L *is at least as conservative than* L' iff for every $\phi \in \mathcal{L}(\mathsf{At})$ and every $v \in \mathcal{V}(\mathsf{At})$, $v^{\mathsf{L}}(\phi) \leq_i v^{\mathsf{L}'}(\phi)$.

As an example, we consider Kleene's logic K.

Kleene's Logic K A 3-valued interpretation v can be extended to arbitrary propositional formulas over At via Kleene semantics (Kleene et al. 1952):

1. $v^{\mathsf{K}}(\neg\phi) = \mathsf{F}$ iff $v^{\mathsf{K}}(\phi) = \mathsf{T}$, $v^{\mathsf{K}}(\neg\phi) = \mathsf{T}$ iff $v^{\mathsf{K}}(\phi) = \mathsf{F}$, and $v^{\mathsf{K}}(\neg\phi) = \mathsf{U}$ iff $v^{\mathsf{K}}(\phi) = \mathsf{U}$;
2. $v^{\mathsf{K}}(\phi \wedge \psi) = \mathsf{T}$ iff $v^{\mathsf{K}}(\phi) = v^{\mathsf{K}}(\psi) = \mathsf{T}$, $v^{\mathsf{K}}(\phi \wedge \psi) = \mathsf{F}$ iff $v^{\mathsf{K}}(\phi) = \mathsf{F}$ or $v^{\mathsf{K}}(\psi) = \mathsf{F}$, and $v^{\mathsf{K}}(\phi \wedge \psi) = \mathsf{U}$ otherwise;
3. $v^{\mathsf{K}}(\phi \vee \psi) = \mathsf{T}$ iff $v^{\mathsf{K}}(\phi) = \mathsf{T}$ or $v^{\mathsf{K}}(\psi) = \mathsf{T}$, $v^{\mathsf{K}}(\phi \vee \psi) = \mathsf{F}$ iff $v^{\mathsf{K}}(\phi) = v^{\mathsf{K}}(\psi) = \mathsf{F}$, and $v^{\mathsf{K}}(\phi \vee \psi) = \mathsf{U}$ otherwise.

Proposition 1. Kleene's Logic K is truth-functional and satisfies semantic relevance.[3]

[1]Notice that $v^{\mathsf{L}}(\alpha) = v^{\mathsf{L}'}(\alpha)$ for any $\alpha \in \mathsf{At}$ and any two three-valued logics L and L'.

[2]Notice that we assume that T is the only designated value. In e.g. paraconsistent logics, also U is taking as a second designated value. However, we stick to the orthodoxy for ADFs and interpret the third truth-value U as "unknown" and therefore not designated.

[3]This follows immediately from the fact that Kleene's logic is *truth-compositional* as defined in e.g. (Chemla and Égré 2019).

2.3 Possibility theory and possibilistic logic

In this subsection, we introduce all necessary preliminaries from possibility theory and possibilistic logic. For more elaborate introductions to possibility theory, we refer the reader to (Dubois and Prade 1993).

Preliminaries from possibility theory Given a set of atoms At, a *possibility distribution* is a mapping $\pi : \Omega(\mathsf{At}) \to [0,1]$. We denote the set of possibility distributions over At by $\mathbf{P}(\mathsf{At})$. π is *normal* if there is some $\omega \in \Omega(\mathsf{At})$ s.t. $\pi(\omega) = 1$. A possibility distribution can be compared using the *principle of minimum specificity* (Dubois and Prade 1986):

Definition 4. Given two possibility distributions π and π', $\pi \leq_s \pi'$ iff $\pi(\omega) \leq \pi'(\omega)$ for every $\omega \in \Omega(\mathsf{At})$.

A possibility distribution induces two measures or degrees that say something about formulas, the *possibility degree* $\Pi_\pi : \mathcal{L}(\mathsf{At}) \to [0,1]$ and the *necessity degree* $\mathcal{N}_\pi : \mathcal{L}(\mathsf{At}) \to [0,1]$. They are defined as follows:

Definition 5. Given a possibility distribution π and a formula $\phi \in \mathcal{L}(\mathsf{At})$:

- $\Pi_\pi(\phi) = \sup\{\pi(\omega) \mid \omega \models \phi\}$.
- $\mathcal{N}_\pi(\phi) = 1 - \Pi_\pi(\neg\phi) = \inf\{1 - \pi(\omega) \mid \omega \models \neg\phi\}$.

Possibilistic logic In (Dubois and Prade 1998), a three-valued logic inspired by possibility theory is presented which is based on defining lower and upper bounds of the evaluation of a formula using a *possibility* and a *necessity measure*. In more detail, given a three-valued interpretation v over At, the set of two-valued interpretations extending a valuation v is defined as $[v]^2 = \{w \in \Omega(\mathsf{At}) \mid v \leq_i w\}$.[4]

Definition 6. Given $v \in \mathcal{V}(\mathsf{At})$, the *necessity measure* $\mathcal{N}_v$ and the *possibility measure* Π_v based on v are functions : $\mathcal{N}_v : \mathcal{L}(\mathsf{At}) \to \{\mathsf{T}, \mathsf{F}\}$ and $\Pi_v : \mathcal{L}(\mathsf{At}) \to \{\mathsf{T}, \mathsf{F}\}$

$$\Pi_v(\phi) = \begin{cases} \mathsf{T} & \text{iff } \omega \models \phi \text{ for some } \omega \in [v]^2 \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$\mathcal{N}_v(\phi) = \begin{cases} \mathsf{T} & \text{iff } \omega \models \phi \text{ for every } \omega \in [v]^2 \\ \mathsf{F} & \text{otherwise} \end{cases}$$

We can now derive a three-valued evaluation $v^{\mathsf{poss}} : \mathcal{L}(\mathsf{At}) \to \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ by stating that:[5]

$$v^{\mathsf{poss}}(\phi) = \begin{cases} \mathsf{T} & \text{iff } \mathcal{N}_v(\phi) = \mathsf{T} \\ \mathsf{U} & \text{iff } \mathcal{N}_v(\phi) = \mathsf{F} \text{ and } \Pi_v(\phi) = \mathsf{T} \\ \mathsf{F} & \text{iff } \mathcal{N}_v(\phi) = \Pi_v(\phi) = \mathsf{F} \end{cases}$$

Example 1. Consider the interpretation v over $\{a, b\}$ with $v(a) = v(b) = \mathsf{U}$. Notice that $\mathcal{N}_v(a \vee \neg a) = \mathsf{T}$ and thus $v^{\mathsf{poss}}(a \vee \neg a) = \mathsf{T}$. However, $\mathcal{N}_v(a \vee b) = \mathcal{N}_v(\neg a) = \mathsf{F}$ and $\Pi_v(a \vee b) = \Pi_v(\neg a) = \mathsf{T}$. Thus, even though $v(a) = v^{\mathsf{poss}}(\neg a) = v(b) = \mathsf{U}$, $v^{\mathsf{poss}}(a \vee b) \neq v^{\mathsf{poss}}(a \vee \neg a)$.

[4] In (Ciucci, Dubois, and Lawry 2014), instead of two-valued interpretations extending a valuation, the notion of *epistemic set* E_v is used, which defined as: $E_v = \{v' \in \Omega \mid v \leq_i v'\}$. It is clear that $E_v = [v]^2$ for any $v \in \mathcal{V}$.

[5] Notice that this enumeration of cases is exhaustive, as for any $v \in \mathcal{V}(\mathsf{At})$ and any $\phi \in \mathcal{L}(\mathsf{At})$, $\mathcal{N}_v(\phi) \leq_{\mathsf{T}} \Pi_v(\phi)$.

Proposition 2. poss is not truth-functional but satisfies *relevance*.

Remark 1. It can be seen that the possibility and necessity measures given a three-valued interpretation v defined in Definition 6 are particular cases of possibility and necessity measures given a possibility distribution π. In more detail, given an interpretation v, set $\pi_v(\omega) = 1$ if $\omega \in [v]^2$ and $\pi_v(\omega) = 0$ otherwise. Then $\Pi_v(\phi) = \mathsf{T}[\mathsf{F}]$ iff $\Pi_{\pi(v)} = 1[0]$ and $\mathcal{N}_v(\phi) = \mathsf{T}[\mathsf{F}]$ iff $\mathcal{N}_{\pi(v)} = 1[0]$. We call the set of possibility distributions $\pi : \Omega(\mathsf{At}) \to \{0,1\}$ the set of *binary possibility distributions*. Clearly, the set of normal binary possibility distributions coincides with $\{\pi_v \mid v \in \mathcal{V}(\mathsf{At})\}$.

2.4 Abstract dialectical frameworks

We briefly recall some technical details on ADFs following loosely the notation from (Brewka et al. 2013). An ADF D is a tuple $D = (\mathsf{At}, L, C)$ where At is a set of statements, $L \subseteq \mathsf{At} \times \mathsf{At}$ is a set of links, and $C = \{C_s\}_{s \in \mathsf{At}}$ is a set of total functions $C_s : 2^{par_D(s)} \to \{\mathsf{T}, \mathsf{F}\}$ for each $s \in \mathsf{At}$ with $par_D(s) = \{s' \in \mathsf{At} \mid (s', s) \in L\}$ (also called acceptance functions). An acceptance function C_s defines the cases when the statement s can be accepted (truth value T), depending on the acceptance status of its parents in D. By abuse of notation, we will often identify an acceptance function C_s by its equivalent *acceptance condition* which models the acceptable cases as a propositional formula. We denote by $\mathfrak{D}(\mathsf{At})$ the set of all ADFs which can be formulated on the basis of At.

Example 2. We consider the following ADF $D_1 = (\{a, b, c\}, L, C)$ with $L = \{(a, b), (b, a), (a, c), (b, c)\}$ and:

$C_a = \neg b \quad C_b = \neg a \quad C_c = \neg a \vee \neg b$

Informally, the acceptance conditions can be read as "a is accepted if b is not accepted", "b is accepted if a is not accepted" and "c is accepted if a is not accepted or b is not accepted".

An ADF $D = (\mathsf{At}, L, C)$ is interpreted through 3-valued interpretations $v \in \mathcal{V}(\mathsf{At})$. The topic of this paper is which logics can be used to extend v to complex formulas in way that is suited for ADFs. Given a set of valuations $V \subseteq \mathcal{V}$, $\sqcap_i V(s) := v(s)$ if for every $v' \in V$, $v(s) = v'(s)$ and $\sqcap_i V(s) = \mathsf{U}$ otherwise. The characteristic operator is defined by $\Gamma_D(v) : \mathsf{At} \to \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ where $s \mapsto \sqcap_i\{w(C_s) \mid w \in [v]^2\}$. Thus, $\Gamma_D(v)$ assigns to s the truth-value that all two-valued extensions of v assign to the condition C_s of s, if they agree on C_s, and U otherwise.

Definition 7. Let $D = (\mathsf{At}, L, C)$ be an ADF with $v : \mathsf{At} \to \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ an interpretation:

- v is a *2-valued model* iff $v \in \Omega(\mathsf{At})$ and $v(s) = v(C_s)$ for every $s \in \mathsf{At}$.
- v is *admissible* for D iff $v \leq_i \Gamma_D(v)$.
- v is *complete* for D iff $v = \Gamma_D(v)$.
- v is *preferred* for D iff v is $\leq_i$-maximal among the admissible interpretations for D.
- v is *grounded* for D iff v is $\leq_i$-minimal among the complete interpretations for D.

We denote by $\mathsf{2mod}(D)$, $\mathsf{admissible}(D)$, $\mathsf{complete}(D)$, $\mathsf{preferred}(D)$, respectively $\mathsf{grounded}(D)$ the sets of 2-valued models and admissible, complete, preferred, respectively grounded interpretations of D.

Example 3 (Example 2 continued)**.** The ADF of Example 2 has three complete models v_1, v_2, v_3 with:
$v_1(a) = \mathsf{T}$ $v_1(b) = \mathsf{F}$ $v_1(c) = \mathsf{T}$
$v_2(a) = \mathsf{F}$ $v_2(b) = \mathsf{T}$ $v_2(c) = \mathsf{T}$
$v_3(a) = \mathsf{U}$ $v_3(b) = \mathsf{U}$ $v_3(c) = \mathsf{U}$

v_3 is the grounded interpretation whereas v_1 and v_2 are preferred interpretations as well as 2-valued models.

3 Logics for ADFs

In this section, we ask the question of which three-valued logics qualify as a logic for ADFs. In particular, given a set of statements At, we will be interested in which logic $\mathcal{V}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \rightarrow \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ can be reasonably said to underlie ADFs. We first recall the notion of a model for ADFs as introduced by (Brewka et al. 2013) and show it is flawed, after which we define models parametrized to a logic. In section 3.2, we show that models parametrized to the logic based on possibility-necessity pairs gives rise to a plausible notion of model. Finally, in section 3.3, we show that there are truth-functional logics that give rise to plausible notions of models, but they commit one to assign determinate truth-values to formulas to which poss assigns the undecided truth-value.

3.1 ADF-models

In (Brewka et al. 2013), *models* are defined as follows:

Definition 8. An interpretation v is a model of an ADF $D = (\mathsf{At}, L, C)$ iff $v(s) \neq \mathsf{U}$ implies $v(s) = v^{\mathsf{K}}(C_s)$ for every $s \in \mathsf{At}$.

In (Brewka et al. 2013) we find the following claim: "Note that admissible interpretations (as well as the special cases complete and preferred interpretations to be defined now) are actually three-valued models." The following example shows that this claim does not hold:

Example 4. $D = (\{a, b\}, L, C)$ with $C_a = b \vee \neg b$ and $C_b = b$. Consider the interpretation v with $v(a) = \mathsf{T}$ and $v(b) = \mathsf{U}$. Since $\sqcap_i [v]^2 (b \vee \neg b) = \mathsf{T}$ and $\sqcap_i [v]^2 (b) = \mathsf{U}$, v is complete. However, $v^{\mathsf{K}}(b \vee \neg b) = \mathsf{U}$ and thus $v(a) \neq v^{\mathsf{K}}(C_a)$, i.e. v is not a model.

One can notice that in (Brewka et al. 2013), Kleene's logic is only used in the definition of models. For all of the other semantics, no reference to Kleene's logic is made. Instead, the Γ_D-operator is used, which makes use of the completions $[v]^2$ of an interpretation v. Thus, models are the only concepts based on Kleene's logic in (Brewka et al. 2013).

We can now generalize the concept of a model by parameterizing it with the underlying logic L as follows:

Definition 9. Given a logic L s.t. $\mathsf{L} : \mathcal{V}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \rightarrow \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ and an ADF D, the set of L-models of D is the set $\mathcal{M}^{\mathsf{L}}(D) = \{v \in \mathcal{V} \mid \text{for every } s \in \mathsf{At} \text{ if } v(s) \neq \mathsf{U} \text{ then } v(s) = v^{\mathsf{L}}(C_s)\}$.

A minimal condition on the set of models is that it includes all the admissible models:

Definition 10. A logic L is *admissible-preserving* if $\mathcal{M}^{\mathsf{L}}(D) \supseteq \mathsf{Admissible}(D)$.

Notice that any admissible-preserving logic L also guarantees that $\mathcal{M}^{\mathsf{L}}(D) \supseteq \mathsf{Sem}(D)$ for any $\mathsf{Sem} \in \{\mathsf{Preferred}, \mathsf{Grounded}, \mathsf{Complete}\}$ since for any Sem-interpretation v, v is admissible.

The following result is a central first insight in the class of admissible-preserving logics:

Lemma 1. A logic L satisfying relevance is admissible-preserving iff $v^{\mathsf{L}}(\phi) \geq_i \sqcap_i [v]^2(\phi)$ for every $v \in \mathcal{V}(\mathsf{At})$ and every $\phi \in \mathcal{L}(\mathsf{At})$.[6]

3.2 Possibilistic logic preserves admissibility

In this section, we show that possibilistic logic poss underlies ADFs. We first show the following crucial lemma, which show that for any interpretation, v^{poss} is identical to $\sqcap_i [v]^2$, the latter being a central technical notion in the semantics of ADFs.

Lemma 2. For any $v \in \mathcal{V}(\mathsf{At})$ and any $\phi \in \mathcal{L}(\mathsf{At})$, $\sqcap_i [v]^2(\phi) = v^{\mathsf{poss}}(\phi)$.

From this Lemma it follows that poss is admissible-preserving:

Proposition 3. Possibilistic logic poss is admissible-preserving.

Furthermore, interestingly enough, the set of models of an ADF under the logic poss collapses to the set of admissible interpretations:

Proposition 4. For any ADF D, $\mathcal{M}^{\mathsf{poss}}(D) = \mathsf{Admissible}(D)$.

Finally, we notice that the Γ_D-function, which is of central importance to the semantics of ADFs, can be easily captured in possibilistic logic. Indeed, for any ADF $D = (\mathsf{At}, L, C)$, $v \in \mathcal{V}(\mathsf{At})$ and $s \in \mathsf{At}$, $\Gamma_D(v)(s) = v^{\mathsf{poss}}(C_s)$ (this is immediate from Lemma 2). From this, it follows that the set of complete models of an ADF $D = (\mathsf{At}, L, C)$ coincides with the following set of interpretations: $\{v \in \mathcal{V}(s) \mid v(s) = v^{\mathsf{poss}}(C_s) \text{ for every } s \in \mathsf{At}\}$.

Remark 2. We draw some consequences from the results above for the case of *abstract argumentation frameworks* (Dung 1995). An abstract argumentation framework is a tuple $(\mathsf{Args}, \rightsquigarrow)$ where Args represents a set of arguments and $\rightsquigarrow \subseteq \mathsf{Args} \times \mathsf{Args}$ is an attack relation between arguments. We denote by $A^+ = \{B \in \mathsf{Args} : B \rightsquigarrow A\}$ the set of attackers of A. It it shown in (Brewka et al. 2013) that argumentation frameworks can be translated in ADFs as follows: given $(\mathsf{Args}, \rightsquigarrow)$, $D(\mathsf{Args}, \rightsquigarrow) = (\mathsf{Args}, \rightsquigarrow, C)$ where $C_A = \bigwedge_{B \in \mathsf{Args}: B \in A^+} \neg B$. Notice that for any $A \in \mathsf{Args}$, C_A is a conjunction of negated literals. For such formulas, Kleene's logic K and Poss coincide, i.e. $v^{\mathsf{K}}(\phi) = v^{\mathsf{Poss}}(\phi)$ for any ϕ built up solely from negated atoms using $\vee$ and $\wedge$ (Ciucci, Dubois, and Lawry 2014,

[6] In view of spatial restrictions, proofs have been left out, but can be found in an online appendix.

Prop. 4.5). It thus follows that for any argumentation framework $(\mathsf{Args}, \rightsquigarrow)$, v is complete iff $v(A) = v^{\mathsf{K}}(C_A)$ for every $A \in \mathsf{Args}$. Likewise, other classes of formulas for which (the non-truth-functional) poss is equivalent to (the truth-functional) K or to other logics, is useful for classes of ADFs, such as bipolar ADFs (Brewka and Woltran 2010; Diller et al. 2020) and ADFs corresponding to logic programs.

3.3 Truth-functional logics

We have shown in the previous section that possibilistic logic underlies ADFs. However, according to Proposition 2, possibilistic logic is not truth-functional. We might therefore ask whether there are some truth-functional three-valued logics that can be seen as a logic for ADFs. A first observation we make is that for any admissible-preserving three-valued logic (truth-functional or otherwise), either the logic coincides with poss or the logic assigns a determinate truth-value T or F to at least one formula ϕ (relative to at least one interpretation v) for which poss evaluates ϕ to U. In other words, poss is the most conservative logic that is admissible-preserving.

Proposition 5. For any admissible preserving logic L, if there is a $\phi \in \mathcal{L}(\mathsf{At})$ and a $v \in \mathcal{V}(\mathsf{At})$ s.t. $v^{\mathsf{L}}(\phi) \neq v^{\mathsf{poss}}(\phi)$, then L is strictly less conservative than poss.

In the rest of this section, we make some observations on what this means for truth-functional logics. To limit our study to a sensible class of three-valued truth-functional logics, we start by making some assumptions on the evaluation of connectives. Firstly, we will assume that any connective conforms with classical logic to determinate truth values, i.e. for any n-ary connective $*$, if $v \in \Omega(\mathsf{At})$, then $v^{\mathsf{L}}(*(\phi_1, \ldots, \phi_n)) = v^{\mathsf{PL}}(*(\phi_1, \ldots, \phi_n))$. Notice that for a truth-functional logic, this means that for every $v \in \mathcal{V}$, $v^{\mathsf{L}}(\phi_i) \in \{\mathsf{T}, \mathsf{F}\}$ for every $1 \leq i \leq n$, implies $v^{\mathsf{L}}(*(\phi_1, \ldots, \phi_n)) = v'^{\mathsf{PL}}(*(\phi_1, \ldots, \phi_n))$ where $v' \in \Omega(\mathsf{At})$ s.t. $v'(\phi_i) = v(\phi_i)$ for every $1 \leq i \leq n$. For conjunction, negation and disjunction this means that every logic has to conform to the following partial truth-tables:

$\wedge$	F	U	T
F	F		F
U			
T	F		T

$\vee$	F	U	T
F	F		T
U			
T	T		T

$\neg$	
F	T
U	
T	F

The full range of possibilities for filling out the $\neg$U-cell of the truth-table for negation results in three negations, known as the involutive $\neg^{\mathsf{i}}$, the paraconsistent $\neg^{\mathsf{p}}$ and the intuitionistic $\neg^{\mathsf{c}}$ (c stands for constructive). These negations have the following truth-tables:

$v(\phi)$	$v(\neg^{\mathsf{i}}\phi)$	$v(\neg^{\mathsf{p}}\phi)$	$v(\neg^{\mathsf{c}}\phi)$
T	F	F	F
U	U	T	F
F	T	T	T

We can show that for any truth-functional logic if the logic is admissible-preserving, it is *strictly* less conservative than poss. We notice that this can be shown without making any assumptions on the connectives other than conformity with classical logic.

Proposition 6. No truth-functional logic L at least as conservative as poss is admissible-preserving.

In passing, we notice that poss also uses an involutive negation, which also implies that $\neg$, in contradistinction to $\vee$ and $\wedge$, is a truth-functional connective in poss.

Fact 1. $\neg$ is a truth-functional, involutive negation under poss.

In the rest of this section, we will further look at which truth-functional logics are exactly admissible-preserving (even though they are strictly less conservative than poss). We shall follow (Ciucci and Dubois 2013) and assume some very basic properties of conjunction, namely (1) $\leq_{\mathsf{T}}$-monotonicity (i.e. if $\mathsf{X} \leq_{\mathsf{T}} \mathsf{Y}$ then $\mathsf{X} \wedge \mathsf{Z} \leq_{\mathsf{T}} \mathsf{X} \wedge \mathsf{Z}$ and $\mathsf{Z} \wedge \mathsf{X} \leq_{\mathsf{T}} \mathsf{Z} \wedge \mathsf{Y}$ for any $\mathsf{X}, \mathsf{Y}, \mathsf{Z} \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$) and (2) Symmetry (i.e. $\mathsf{U} \star \mathsf{T} = \mathsf{T} \star \mathsf{U}$). This results in the following partial truth-table:

$\wedge$	F	U	T
F	F	F	F
U	F		
T	F		T

In the rest of this section, we determine which truth-functional logics with a conjunction as defined above are admissible preserving, by systematically studying all options for the cells $\mathsf{U} \wedge \mathsf{U}$, $\mathsf{U} \wedge \mathsf{T}$ and $\mathsf{T} \wedge \mathsf{U}$.

Involutive negation We show that no truth-functional logic based on an involutive negation is admissible-preserving. Intuitively, the reason is that any such logic is strictly more conservative than poss. A particularly relevant example of this is a tautology like $a \vee \neg a$, which is evaluated to U by any truth-functional logic based on an involutive negation if $v(a) = \mathsf{U}$.

Proposition 7. There exists no truth-functional logic L with an involutive negation that is admissible-preserving.

Paraconsistent negation When we look at truth-functional logics using a paraconsistent negation (and a $\leq_{\mathsf{T}}$-monotonic conjunction conformant with classical logic), a logic can only be admissible-preserving if it makes use of the conjunction known as Bochvar's external conjunction (Bochvar and Bergmann 1981) and which we denote by $\wedge^{\mathsf{B}}$. As disjunction, we use $\vee^1$ (defined below). These connectives have the following truth-tables:

$\wedge^{\mathsf{B}}$	F	U	T
F	F	F	F
U	F	F	F
T	F	F	T

$\vee^1$	F	U	T
F	F	U	T
U	U	U	T
T	T	T	T

The main theorem of this section expresses that there exists a truth-functional three-valued logic using a paraconsistent negation that is admissible-preserving, but it is strictly less conservative than poss. Notice that the fact that this logic is strictly less conservative than poss follows immediately from Proposition 6: the main positive result here is that there exists a truth-functional three-valued logic using a paraconsistent negation that is admissible-preserving. Since the goal of Proposition 8 is to show merely that an admissible-preserving logic based on paraconsistent negation exists, no particular motivation for the choice of conjunction and disjunction is needed, besides the fact that it fulfils some basic properties like $\geq_{\mathsf{T}}$-monotonicity and symmetry (and similarly for Proposition 9).

Proposition 8. $\mathsf{L}^{\neg^{\mathsf{p}},\wedge^{\mathsf{B}},\vee^1}$ is admissible-preserving and strictly less conservative than poss.

Intuitionistic negation For an intuitionistic negation, we can show similarly to the previous section that there is a logic which is admissible-preserving (but again less conservative than poss). With regards to disjunction, note that conformity with v^{poss} requires that $v(\mathsf{U} \vee \mathsf{F}) = v(\mathsf{F} \vee \mathsf{U}) = \mathsf{T}$ to ensure that e.g. $v^{\mathsf{L}}(a \vee \neg a) = \mathsf{T}$ even when $v(a) = \mathsf{U}$. The other cells of the truth-table for disjunction can then be filled in using conformity to classical logic and left- and right-monotonicity. We shall use here the conjunction known as Sette's conjunction (Sette 1973). This is, in fact, not the only conjunction that can be used (even though $\wedge^{\mathsf{B}}$ would not result in an admissible-preserving logic).[7] The truth-tables for $\wedge^{\mathsf{S}}$ is written out below. We shall use for a disjunction $\vee^2$ as defined below:

$\wedge^{\mathsf{S}}$	F	U	T
F	F	F	F
U	F	T	T
T	F	T	T

$\vee^2$	F	U	T
F	F	T	T
U	T	T	T
T	T	T	T

We can now show the main result of this section:

Proposition 9. $\mathsf{L}^{\neg^{\mathsf{c}},\wedge^{\mathsf{S}},\vee^2}$ is admissible preserving and strictly less conservative than poss.

4 Strong equivalence

Strong equivalence (Lifschitz, Pearce, and Valverde 2001) is a notion of equivalence for non-monotonic formalisms which states that two knowledge bases (in this case, ADFs) are strongly equivalent if after the addition of any new information, the knowledge bases are equivalent (i.e. the semantics coincide). On the basis of our characterisation results in Section 3.2, one might expect to derive characterisations of strong equivalence for ADFs. After all, in Section 3.2 we have shown that possibilistic logic is a logic underlying abstract dialectical argumentation. Indeed, our results can be used to derive a characterisation of strong equivalence for ADFs. In more detail, we show that strong equivalence for ADFs coincides with pairwise equivalence of acceptance conditions under classical logic. Given our results from Section 3.2, this is not surprising, as equivalence under classical logic coincides with possibilistic logic:

Proposition 10. For any $\phi, \psi \in \mathcal{L}(\mathsf{At})$, $\mathcal{V}^{\mathsf{poss}}(\phi) = \mathcal{V}^{\mathsf{poss}}(\psi)$ iff ϕ and ψ are PL-equivalent (i.e. $[\phi] = [\psi]$).

We first elucidate the concept of strong equivalence for ADFs in more detail. Recall that a central concept in the definition of strong equivalence is *the addition of knowledge*. For many formalisms, addition of knowledge can be modelled using set-theoretic union. For ADFs, this is not feasible for several reasons. Firstly, simply combining two ADFs under set-theoretic union does, rather evidently, not result in a new ADF but rather in a set of ADFs. Secondly, one has to ensure that one models appropriately the combination of two ADFs with shared atoms. Consider e.g. two ADFs $D_1 = (\{a\}, L_1, C_a^1)$ and $D_2 = (\{a\}, L_2, C_a^2)$ with $C_a^1 = a$ and $C_a^2 = \neg a$. Clearly, the combination of ADFs has to be modelled on the basis of some logical operator combining C_a^1 and C_a^2 in a single new condition C_a. We specify a general model of addition of ADFs which allows for the combination of conditions using either disjunction or conjunction. Given a set of atoms At, an *and-or-assignment for* At is a mapping $\odot : \mathsf{At} \to \{\wedge, \vee\}$. Intuitively, an and-or-assignment specifies for every atom $s \in \mathsf{At}$ whether conditions for s will be combined using $\wedge$ or using $\vee$. Based on an and-or-assignment $\odot$, we can now define the combination of two ADFs using $\odot$:

Definition 11. [8] Let $D_1 = (\mathsf{At}_1, L_1, C_1)$ and $D_2 = (\mathsf{At}_2, L_2, C_2)$ be two ADFs and $\odot$ an and-or-assignment for At. Define $D_1 \Cup_{\odot} D_2 = (\mathsf{At}_1 \cup \mathsf{At}_2, L_1 \cup L_2, C^{\odot})$ with and $C^{\odot} = \{C_s^{\odot}\}_{s \in \mathsf{At}}$, where:

$$C_s^{\odot} = \begin{cases} C_s^1 \odot(s) C_s^2 & \text{if } s \in \mathsf{At}_1 \cap \mathsf{At}_2 \\ C_s^1 & \text{if } s \in \mathsf{At}_1 \setminus \mathsf{At}_2 \\ C_s^2 & \text{if } s \in \mathsf{At}_2 \setminus \mathsf{At}_1 \end{cases}$$

Example 5. Consider D as in Example 2, $D' = (\{a, b, d\}, L', C)$ with $C_a = b$, $C_b = d \wedge \neg a$ and $C_d = \neg a$, and $\odot(a) = \odot(b) = \wedge$ and $\odot(c) = \odot(d) = \vee$. Then $D_1 \Cup_{\odot} D_2 = (\{a, b, c, d\}, L_1 \cup L_2, C^{\odot})$ where:

$$C_a^{\odot} = \neg b \wedge b \qquad C_b^{\odot} = \neg a \wedge d \wedge \neg a$$
$$C_c^{\odot} = \neg a \vee \neg b \qquad C_d^{\odot} = \neg a$$

We now define strong equivalence for ADFs as follows:

Definition 12. Two ADFs $D_1 = (\mathsf{At}, L_1, C_1)$ and $D_2 = (\mathsf{At}, L_2, C_2)$ are strongly equivalent under semantics Sem iff for any $D \in \mathfrak{D}(\mathsf{At})$ and any and-or-assignment $\odot$ for At, $\mathsf{Sem}(D_1 \Cup_{\odot} D) = \mathsf{Sem}(D_2 \Cup_{\odot} D)$.

For any of the admissible, complete, preferred and grounded semantics, pairwise equivalence of conditions under classical logic is a sufficient and necessary condition for strong equivalence:

Proposition 11. Let some $\mathsf{Sem} \in \{\mathsf{Admissible}, \mathsf{Complete}, \mathsf{Preferred}, \mathsf{Grounded}\}$ and two ADFs $D_1 = (\mathsf{At}, L_1, C_1)$ and $D_2 = (\mathsf{At}, L_2, C_2)$ be given. Then: for every $s \in \mathsf{At}$, $C_1^s \equiv_{\mathsf{PL}} C_2^s$ iff D_1 and D_2 are strongly equivalent under semantics Sem.

Interestingly enough, if we restrict the and-or-assignments allowed in combinations of ADFs, our result above does not hold anymore. In particular, for $\oplus \in \{\vee, \wedge\}$, we say that D_1 and D_2 are $\oplus$-strongly

[7] To see this, observe that then e.g. $v(a) = v(b) = \mathsf{U}$ would set $v^{\mathsf{L}}((a \wedge b) \vee (\neg a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge \neg b)) = \mathsf{F}$ even though $v^{\mathsf{poss}}((a \wedge b) \vee (\neg a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge \neg b)) = \mathsf{T}$, contradicting Lemma 1 and the assumption that L is admissible-preserving.

[8] Our notion of composition of ADFs is clearly a generalization of that of (Gaggl and Strass 2014).

equivalent if for any $D \in \mathfrak{D}(\mathsf{At})$ and any and-or-assignment $\odot$ for At for which $\odot(s) = \oplus$ for any $s \in \mathsf{At}$, $\mathsf{Sem}(D_1 \Cup_\odot D) = \mathsf{Sem}(D_2 \Cup_\odot D)$.

Proposition 12. Let some $\mathsf{Sem} \in \{\mathsf{Admissible}, \mathsf{Complete}, \mathsf{Preferred}, \mathsf{Grounded}\}$ and some $\oplus \in \{\vee, \wedge\}$ be given. Then there exist $\oplus$-strongly equivalent (under Sem) ADFs $D_1 = (\mathsf{At}, L_1, C_1)$ and $D_2 = (\mathsf{At}, L_2, C_2)$ for which for some $s \in \mathsf{At}$, $C_1^s \not\equiv_{\mathsf{PL}} C_2^s$.

Proof. We show the claim for $\odot = \wedge$. Consider the ADFs $D_1 = (\{a, b, c\}, L, C^1)$ and $D_2 = (\{a, b, c\}, L, C^2)$ with:

$$\begin{array}{llll} C^1_a = & \bot & C^2_a = & \bot \\ C^1_b = & \bot & C^2_b = & \bot \\ C^1_c = & \neg a \wedge b \wedge c & C^2_c = & a \wedge \neg b \wedge c \end{array}$$

Notice that $C^1_c \not\equiv_{\mathsf{PL}} C^2_c$. We show that for any $D_3 = (\{a, b, c\}, L, C^3)$, $\mathsf{Admissible}(D_1 \otimes D_3) = \mathsf{Admissible}(D_2 \otimes D_3)$. Indeed, notice first that for any $\phi \in \mathcal{L}(\{a, b, c\})$, any $1 \leq i \leq 2$ and any $x \in \{a, b\}$, $\sqcap[v]^2(C^i_x \wedge \phi) = \mathsf{F}$. Thus, if $v \in \mathsf{Admissible}(D_1 \otimes D_3)$, $v(x) \leq_i \mathsf{F}$ for any $x \in \{a, b\}$. For any such v, $\sqcap[v]^2(\neg a \wedge b \wedge c \wedge \phi) \in \{\mathsf{U}, \mathsf{F}\}$ and $\sqcap[v]^2(a \wedge \neg b \wedge c \wedge \phi) \in \{\mathsf{U}, \mathsf{F}\}$. Thus, for $1 \leq i \leq 2$, if $v \in \mathsf{Admissible}(D_i \otimes D_3)$, $v(c) \leq_i \mathsf{F}$. Suppose now first that $v(c) = \mathsf{U}$. Then $v(c) \leq_i v(C^2_c \wedge \phi)$ and thus $v \in \mathsf{Admissible}(D_1 \otimes D_3)$. If $v(c) = \mathsf{F}$, then clearly $\sqcap[v]^2(\neg a \wedge b \wedge c \wedge \phi) = \sqcap[v]^2(a \wedge \neg b \wedge c \wedge \phi) = \mathsf{F}$. Otherwise, $\sqcap[v]^2(\neg a \wedge b \wedge c \wedge \phi) \geq_i v(c)$ and $\sqcap[v]^2(a \wedge \neg b \wedge c \wedge \phi) \geq_i v(c)$. Thus, $v \in \mathsf{Admissible}(D_1 \otimes D_3)$ implies $v \in \mathsf{Admissible}(D_2 \otimes D_3)$. By symmetry we obtain $\mathsf{Admissible}(D_1 \otimes D_3) = \mathsf{Admissible}(D_2 \otimes D_3)$. The proof for other semantics is similar.

To show the claim for $\odot = \oplus$, a similar counter-example can be constructed. □

We leave the further investigation of such weaker notions of strong equivalence for future work.

5 ADFs from the perspective of possibility Theory

We now look further into the perspective offered by possibility theory on ADFs. In more detail, based on the strong connection established between ADFs and possibilistic logic (Sec. 3.2), we unpack the semantics of ADFs using concepts known from possibility theory. This will allow us to straightforwardly formulate generalizations of ADFs. We first show how all semantic concepts from abstract dialectical argumentation correspond to notions from possibility theory. Thereafter, we use these correspondences to define *possibilistic ADFs*.

5.1 ADFs interpreted in possibility theory

In this section we interpret the semantics of ADFs in terms of possibility theory, and generalize the semantics of ADFs to possibility distributions.

We start by looking closer at the information ordering. Recall that one interpretation v is less or equally informative than v' iff v' assigns the same determinate truth-value to every atom s for which v assigns a determinate truth-value. It turns out that this is equivalent to requiring that:

$$\mathcal{N}_v(s) \leq \mathcal{N}_{v'}(s) \text{ and } \Pi_v(s) \geq \Pi_{v'}(s) \text{ for every } s \in \mathsf{At}$$

or, equivalently:

$$\Pi_v(\overline{s}) \geq \Pi_{v'}(\overline{s}) \text{ and } \Pi_v(s) \geq \Pi_{v'}(s) \text{ for every } s \in \mathsf{At}$$

Fact 2. For any $v, v' \in \mathcal{V}$, $v \leq_i v'$ iff $\Pi_v(\overline{s}) \geq \Pi_{v'}(\overline{s})$ and $\Pi_v(s) \geq \Pi_{v'}(s)$ for every $s \in \mathsf{At}$.[9]

From this relation, we can derive that $\leq_s$ and $\leq_i$ are each-others converses when we look at three-valued interpretations (or equivalently, normal binary possibility distributions):

Proposition 13. For any interpretations $v, v' \in \mathcal{V}(\mathsf{At})$, $v \leq_i v'$ iff $\pi_{v'} \leq_s \pi_v$.

Based on Fact 2, we can define the information-ordering $\leq_i$ over the set of possibility distributions $\mathbf{P}(\mathsf{At})$ as follows: $\pi \leq_i \pi'$ iff $\Pi_\pi(\overline{s}) \geq \Pi_{\pi'}(\overline{s})$ and $\Pi_\pi(s) \geq \Pi_{\pi'}(s)$ for every $s \in \mathsf{At}$. In other words, more informative possibility distributions assign lower possibility measures to literals. This might seem at first counter-intuitive, when rephrased in terms of the dual necessity measures, the intuition becomes clearer:

$$\pi \leq_i \pi' \text{ iff } \mathcal{N}_\pi(\overline{s}) \leq \mathcal{N}_{\pi'}(\overline{s}) \text{ and } \mathcal{N}_\pi(s) \leq \mathcal{N}_{\pi'}(s)\ \forall s \in \mathsf{At}$$

Proposition 13 only generalizes to the setting of possibility distributions in one direction: indeed $\leq_i$ as defined over possibility distributions is a generalization of the reverse specificity-ordering:

Fact 3. For some possibility distributions $\pi, \pi' \in \mathbf{P}(\mathsf{At})$, $\pi \leq^s \pi'$ implies $\pi' \leq_i \pi$.

The following examples shows that the reverse direction of Proposition 13 does not generalize to the case of arbitrary normal possibility distributions:

Example 6. Consider the following possibility distributions $\pi, \pi' \in \mathbf{P}(\{a, b\})$:

ω	$\pi(\omega)$	$\pi'(\omega)$	ω	$\pi(\omega)$	$\pi'(\omega)$
ab	1	1	$a\overline{b}$	0.1	1
$\overline{a}b$	1	0.1	$\overline{a}\,\overline{b}$	1	1

Notice that $\Pi_\pi(s) = \Pi_{\pi'}(s)$ for any literal s and thus $\pi \leq_i \pi'$ and $\pi' \leq_i \pi$. However, π and π' are $\leq_s$ incomparable, as $\pi(a\overline{b}) \leq \pi'(a\overline{b})$ and $\pi(\overline{a}b) \leq \pi'(\overline{a}b)$. This shows that Proposition 13 does not generalize from $\mathcal{V}(\mathsf{At})$ to $\mathbf{P}(\mathsf{At})$.

We now characterize admissible and complete interpretations in terms of possibility and necessity measures. Admissible interpretations correspond to possibility distributions for which every node s has:

- a degree of necessity equal or less than the degree of necessity of the corresponding condition C_s; and
- a degree of possibility equal or higher than the degree of possibility of the corresponding condition C_s.

[9] Recall that $\leq_s$ is defined in Definition 4.

In other words, the interval formed by the degree of possibility and necessity of C_s is a sub-interval of the correspondent interval for s.

Completeness strengthens this by requiring the necessity degree, respectively the possibility degree, of a node to be equal to the corresponding degree of its condition.

Proposition 14. Given an ADF $D = (\mathsf{At}, L, C)$ and an interpretation $v \in \mathcal{V}(\mathsf{At})$:

1. v is admissible iff for every $s \in \mathsf{At}$, $\mathcal{N}_v(s) \leq \mathcal{N}_v(C_s)$ and $\Pi_v(s) \geq \Pi_v(C_s)$ (or, equivalently $\Pi_v(\neg s) \geq \Pi_v(\neg C_s)$ and $\Pi_v(s) \geq \Pi_v(C_s)$).
2. v is complete iff for every $s \in \mathsf{At}$, $\mathcal{N}_v(s) = \mathcal{N}_v(C_s)$ and $\Pi_v(s) = \Pi_v(C_s)$ (or, equivalently $\Pi_v(\neg s) = \Pi_v(\neg C_s)$ and $\Pi_v(s) = \Pi_v(C_s)$).

We can now straightforwardly generalize the ADF semantics to possibility distributions:

Definition 13. Given an ADF $D = (\mathsf{At}, L, C)$ and a normal possibility distribution $\pi \in \mathbf{P}(\mathsf{At})$:

- π is *admissible (for D)* iff $\Pi_\pi(\neg s) \geq \Pi_\pi(\neg C_s)$ and $\Pi_\pi(s) \geq \Pi_\pi(C_s)$ for every $s \in \mathsf{At}$.
- π is *complete (for D)* iff $\Pi_\pi(\neg s) = \Pi_\pi(\neg C_s)$ and $\Pi_\pi(s) = \Pi_\pi(C_s)$ for every $s \in \mathsf{At}$.
- π is *grounded (for D)* iff π is a $\leq_i$-minimal complete possibility distribution.
- π is *preferred (for D)* iff π is a $\leq_i$-maximal admissible possibility distribution.

We can show that these semantics satisfy the following basic argumentative properties for ADFs:

Proposition 15. Given an ADF $D = (\mathsf{At}, L, C)$: (1) there exists a unique grounded possibility distribution for π; (2) any preferred possibility distribution for π is complete.

The above proposition is shown by defining a function $\mathfrak{G}_D : \mathbf{P}(\mathsf{At}) \to \mathbf{P}(\mathsf{At})$ that returns, for a possibility distribution π, a new possibility distribution $\mathfrak{G}_D(\pi)$ s.t. for any $s \in \mathsf{At}$, $\Pi_{\mathfrak{G}_D}(\pi)(s) = \Pi_\pi(C_s)$ and $\Pi_{\mathfrak{G}_D}(\pi)(\overline{s}) = \Pi_\pi(\overline{C_s})$. To define such a $\mathfrak{G}_D$-function constructively, we need some preliminaries first. Given a set of formulas $\{\phi_1, \ldots, \phi_n\}$ and a possibility measure $\pi \in \mathbf{P}(\mathsf{At})$, we call the *possibility-vector of* $\{\phi_1, \ldots, \phi_n\}$ *given* π the vector $\langle \dot{\phi_{i_1}}, \ldots, \dot{\phi_{i_1}} \rangle$ s.t. for every $1 \leq i \leq n$, ϕ_i and $\overline{\phi_i}$ both occur exactly once in the vector and the vector is arranged w.r.t. ascending degree of possibility, i.e. for $j \leq k$ it holds that $\Pi_\pi(\dot{\phi_{i_j}}) \leq \Pi_\pi(\dot{\phi_{i_k}})$. We can now define the $\mathfrak{G}_D$-function as follows:

Definition 14. Let a possibility distribution $\pi \in \mathbf{P}(\mathsf{At})$, an ADF $D = (\mathsf{At}, L, C)$, and the possibility-vector $\langle \dot{C}_{s_{i_1}}, \ldots, \dot{C}_{s_{i_k}} \rangle$ of $\{C_{s_1}, \ldots, C_{s_n}\}$ given π be given. Then we define $\mathfrak{G}_D(\pi)$ as the possibility distribution s.t. $\mathfrak{G}_D(\pi)(\omega) = \sup_\pi([\dot{C}_{s_{i_j}}])$ for every $\omega \in [\dot{s_{i_j}}] \setminus \bigcup_{l=1}^{j-1} [\dot{s_{i_l}}]$ for every $1 \leq j \leq k$.[10]

[10]This construction has been implemented in Java using the `Tweety`-library. The implementation can be found online.

Thus, $\mathfrak{G}_D(\pi)$ is constructed iteratively, starting with the literal $\dot{s}$ for which $\Pi_\pi(\dot{C}_s)$ is the lowest among all literals. For all worlds satisfying $\dot{s}$, we set $\mathfrak{G}_D(\pi)(\omega) = \Pi_\pi(\dot{C}_s)$. Then, we take the second element $\dot{s}'$ of the possibility-vector, and proceed similarly for all worlds satisfying $\dot{s}'$ but not satisfying $\dot{s}$. This process is repeated for all elements of the possibility-vector.

Example 7. Let $D_2 = (\{a, b, c\}, L, C)$ with:

$$C_a = \neg b \wedge \neg c \quad C_b = \neg a \quad C_c = c$$

and consider π_1 defined by:

ω	$\pi_1(\omega)$	ω	$\pi_1(\omega)$	ω	$\pi_1(\omega)$	ω	$\pi_1(\omega)$
abc	0.1	$ab\overline{c}$	0.2	$a\overline{b}c$	0.3	$a\overline{b}\overline{c}$	1.0
$\overline{a}bc$	0.3	$\overline{a}b\overline{c}$	0.2	$\overline{a}\overline{b}c$	0.1	$\overline{a}\overline{b}\overline{c}$	0.1

π gives rise to the following possibility measures for acceptance conditions and their negation:

ϕ	C_a	$\neg C_a$	C_b	$\neg C_b$	C_c	$\neg C_c$
$\Pi_\pi(\phi)$	1.0	0.3	0.3	1.0	0.3	1.0

This results in the following possibility-vector for D given π: $\langle \overline{C_a}, C_b, C_c, \overline{C_b}, C_a, \overline{C_c} \rangle$.

Since $\overline{C_a}$ occurs first in the possibility-vector, we set $\mathfrak{G}_D(\pi)(\overline{a}bc) = \mathfrak{G}_D(\pi)(\overline{a}b\overline{c}) = \mathfrak{G}_D(\pi)(\overline{a}\overline{b}c) = \mathfrak{G}_D(\pi)(\overline{a}\overline{b}\overline{c}) = 0.3$. Since $\Pi_\pi(C_b) = \Pi_\pi(C_c)$, we proceed similarly with all worlds that satisfy c or b, i.e. $\mathfrak{G}_D(\pi)(abc) = \mathfrak{G}_D(\pi)(ab\overline{c}) = \mathfrak{G}_D(\pi)(a\overline{b}c) = 0.3$.

Then, we proceed to the next element of the possibility-vector, $\overline{C_b}$, and, since $\Pi_\pi(\overline{C_b}) = 1.0$, we set $\mathfrak{G}_D(\pi)$ for every world that satisfies $\overline{b}$ but does not satisfy $\overline{a}$, c or b (i.e. every element of $[\overline{b}] \setminus ([\overline{a}] \cup [c] \cup [b])$) to 1.0. Thus, $\mathfrak{G}_D(\pi)(a\overline{b}\overline{c}) = 1.0$. Since every world in $\Omega(\mathsf{At})$ has been assigned a value, the construction of $\mathfrak{G}_D(\pi)$ is finished.

The $\mathfrak{G}_D$-function is a faithful generalization of the Γ_D-operator (in view of Remark 1):

Proposition 16. For any ADF D and any three-valued interpretation $v \in \mathcal{V}(\mathsf{At})$, $\Pi_{\Gamma_D(v)}(s) = \mathsf{T}[\mathsf{F}]$ iff $\Pi_{\mathfrak{G}_D(\pi_v)}(s) = 1[0]$ and $\mathcal{N}_{\Gamma_D(v)}(s) = \mathsf{T}[\mathsf{F}]$ iff $\mathcal{N}_{\mathfrak{G}_D(\pi_v)}(s) = 1[0]$.

Thus, the information order, as well as the semantics of ADFs can all be straightforwardly rephrased using possibility measures Π and necessity measures $\mathcal{N}$. On the basis of this interpretation, the semantics for ADFs were generalized from three-valued interpretations – which can be viewed as binary possibility distributions) – to arbitrary possibility distributions. In the next section, we use this generalization to define *possibilistic* ADFs.

5.2 Possibilistic ADFs

We now introduce possibilistic ADFs as a a quantitative extension of ADFs, which can assign a degree of plausibility to the acceptance of nodes. This allows, among others, the incorporation of possibilistic constraints on nodes and their acceptance condition.

Definition 15. An *ADF with possibilistic constraints* (pADF) is a tuple $\mathfrak{D} = (\mathsf{At}, L, C, \rho)$ where (At, L, C) is an ADF and $\rho : \mathsf{At} \to [0, 1]$.

The intuitive interpretation of ρ_S is that they form an upper limit on the possibility of the nodes of an pADF.

Example 8. Consider the following pADF:

$$\mathfrak{D} = (\{a, b, c\}, L, \{C_a = \neg b \wedge \neg c, C_b = \neg a, C_c = c\}, \{\rho(a) = 1, \rho(b) = 0.8, \rho(c) = 0.4\})$$

Definition 16. Given a pADF $\mathfrak{D} = ((\mathsf{At}, L, C, \rho)$, a normal possibility distribution $\pi : S \to [0, 1]$ is:

- p-*permissible (for* $\mathfrak{D}$*)* iff $\Pi_\pi(s) \leq \rho(s)$ for every $s \in \mathsf{At}$.
- p-*admissible (for* $\mathfrak{D}$*)* iff it is admissible and p-permissible for $\mathfrak{D}$.
- p-*complete (for* $\mathfrak{D}$*)* iff it is complete and p-permissible for $\mathfrak{D}$.
- p-*grounded (for* $\mathfrak{D}$*)* if it is $\leq_i$-least specific p-complete interpretation for $\mathfrak{D}$.
- p-*preferred (for* $\mathfrak{D}$*)* if it is a $\leq_i$-maximal p-admissible interpretation for $\mathfrak{D}$.

Example 9. The following possibility distributions is p-grounded for the pADF $\mathfrak{D}$ from Example 8:

ω	$\pi_1(\omega)$	ω	$\pi_1(\omega)$	ω	$\pi_1(\omega)$	ω	$\pi_1(\omega)$
abc	0.4	$ab\overline{c}$	0.8	$a\overline{b}c$	0.4	$a\overline{b}\overline{c}$	1
$\overline{a}bc$	0.4	$\overline{a}b\overline{c}$	0.8	$\overline{a}\overline{b}c$	0.4	$\overline{a}\overline{b}\overline{c}$	0.8

The following distributions is p-preferred for $\mathfrak{D}$:

ω	$\pi_2(\omega)$	ω	$\pi_2(\omega)$	ω	$\pi_2(\omega)$	ω	$\pi_2(\omega)$
abc	0	$ab\overline{c}$	0	$a\overline{b}c$	0	$a\overline{b}\overline{c}$	1
$\overline{a}bc$	0	$\overline{a}b\overline{c}$	0	$\overline{a}\overline{b}c$	0	$\overline{a}\overline{b}\overline{c}$	0

Notice that the grounded possibility distribution for $D = (\{s, c\}, L, \{C_s = \neg c, C_c = \neg s\})$ is *not* p-complete for $\mathfrak{D}$. Indeed, the grounded extension for D is given by $\pi_3(\omega) = 1$ for every $\omega \in [\{a, b, c\}]$. To see that π_3 is not p-complete for $\mathfrak{D}$, it suffices to observe that $\Pi_{\pi_3}(b) = 1 > \rho(b) = 0.8$.

We remark here that there might not exist a unique p-grounded extension for a given pADF. Furthermore, there might be pADFs for which there do not exist even p-admissible extensions. For example, if we change $\rho(a) = 0.9$ in the pADF from Example 8 there does exist a normal p-admissible possibility distribution. A pADF for which no p-admissible extensions exist can be seen as faultily specified model. This is not unlike epistemic approaches to probablistic argumentation (Hunter and Thimm 2017), where certain requirements such as coherence w.r.t. an argumentation framework are required in order to ensure a good fit between a probability function and an argumentation framework (Hunter and Thimm 2017). We leave the investigation of such requirements for pADFs for future work.

6 Related work

In this paper, we have investigated three-valued monotonic logics underlying ADFs. To the best of our knowledge, this work is the first systematic such study, but there are some works which contain some similar results or questions. In (Baumann and Heinrich 2020), it is shown that there is no truth-functional three-valued logic L s.t. for every $v \in \mathcal{V}(\mathsf{At})$ and every $\phi \in \mathcal{L}(\mathsf{At})$, $v^{\mathsf{L}}(\phi) = \sqcap_i [v]^2(\phi)$. Lemma 1 is a generalization of this result. Our paper continues where (Baumann and Heinrich 2020) stopped, since we show which truth-functional logics are admissible-preserving, and that there is a non-truth-functional monotonic three-valued logic, poss for which $v^{\mathsf{poss}}(\phi) = \sqcap_i [v]^2(\phi)$ for every $v \in \mathcal{V}(\mathsf{At})$ and every $\phi \in \mathcal{L}(\mathsf{At})$. In (Heyninck and Kern-Isberner 2020) ADFs are translated in autoepistemic logic via epistemic models, which are related to possibilistic logic (Ciucci and Dubois 2012).

With respect to the possibilistic ADFs introduced in this paper, we make a comparison with *weighted ADFs* (Brewka et al. 2018). Weighted ADFs generalize ADFs by allowing interpretations which map nodes to elements of V_{U}, which is a complete partial order constructed on the basis of a chosen set V of values combined with the U-value, which forms the $\leq_i$-least element under the information order over V_{U}. This is a very general model of weighted argumentation, which possibilistic ADFs cannot lay claim to. On the other hand, in possibilistic ADFs, there is no need to postulate an additional value U, since it arises naturally from the possiblistic semantics as a discrepancy between the necessity measure $\mathcal{N}$ and the possibility measure Π. (Wu et al. 2016) defines *fuzzy argumentation frameworks*, where arguments and attacks are assigned a degree of belief. The central concept in this work is the concept of a *tolerable attack* which is an attack such that the belief in the attacked argument is not greater than the composition (according to an appropriate composition operator such as the Gödel t-norm) of the belief in the attacking argument and the belief in the attack. Argumentation semantics can then be defined using this concept of weakening attack. (Janssen, De Cock, and Vermeir 2008) uses a similar semantics. It can be seen that these semantics are dependent on the syntactical structure of argumentation frameworks consisting of arguments and attacks. Furthermore, it should be noticed that even though possiblistic logic is related to fuzzy logic, they are far from equivalent. Among the most poignant differences between these two formalisms in our setting is probably truth-functionality. For example, given the fuzzy degree of belief in two formulas ϕ_1 and ϕ_2, one can exactly determine the fuzzy degree of belief in $\phi_1 \wedge \phi_2$, whereas based on the possibility measure assigned to ϕ_1 and ϕ_2 according to π, one can merely determine an upper bound $min\{\Pi_\pi(\phi_1), \Pi_\pi(\phi_2)\}$ on $\Pi_\pi(\phi_1 \wedge \phi_2)$.

7 Conclusion

In this paper, we have investigated monotonic three-valued logics that underlie abstract dialectical argumentation. The central result is that possibilistic logic is closely related to abstract dialectical argumentation, as it is the most conservative admissible-preserving logic, and allows to straightforwardly codify all central semantical notions from abstract dialectial argumentation. We have also exhaustively investigated the ADF-related properties of truth-functional three-valued logics, showing that truth-functional logics using involutive negation are not admissible-preserving, whereas there exist admissible-preserving truth-functional logics using an intuitionistic or paraconsistent negation, but these are strictly less conservative than possibilistic logic. Furthermore, we have illustrated the fruitfulness of our results by (1) characterising strong equivalence and (2) proposing *possibilistic ADFs*, which allow for quantitative reasoning in ADFs in a way that faithfully generalizes (qualitative) reasoning in ADFs. We believe that the connection between possibilistic logic and possibility theory on the one hand, and (abstract) argumentation and ADFs on the other hand, will provide a useful tool for work argumentation, by providing opportunities for the application of results and insights from possibility theory in argumentation.

Acknowledgements

The research reported here was supported by the Deutsche Forschungsgemeinschaft under grant KE 1413/11-1. We thank Jonas Schumacher for helpful comments.

References

Baumann, R., and Heinrich, M. 2020. Timed abstract dialectical frameworks: A simple translation-based approach. *Computational Models of Argument: Proceedings of COMMA* 103 – 110.

Bochvar, D. A., and Bergmann, M. 1981. On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus. *History and Philosophy of Logic* 2(1-2):87–112.

Brewka, G., and Woltran, S. 2010. Abstract dialectical frameworks. In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning*.

Brewka, G.; Ellmauthaler, S.; Strass, H.; Wallner, J. P.; and Woltran, S. 2013. Abstract dialectical frameworks revisited. In *Proceedings of the 23rd International Joint Conference on Artificial Intelligence (IJCAI'13)*.

Brewka, G.; Strass, H.; Wallner, J.; and Woltran, S. 2018. Weighted abstract dialectical frameworks. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32.

Cayrol, C., and Lagasquie-Schiex, M.-C. 2005. On the acceptability of arguments in bipolar argumentation frameworks. In *European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty*, 378–389. Springer.

Chemla, E., and Égré, P. 2019. Suzko's problem: Mixed consequence and compositionality. *The Review of Symbolic Logic* 12(4):736–767.

Ciucci, D., and Dubois, D. 2012. Three-valued logics for incomplete information and epistemic logic. In *European Workshop on Logics in Artificial Intelligence*, 147–159. Springer.

Ciucci, D., and Dubois, D. 2013. A map of dependencies among three-valued logics. *Information Sciences* 250:162–177.

Ciucci, D.; Dubois, D.; and Lawry, J. 2014. Borderline vs. unknown: comparing three-valued representations of imperfect information. *International Journal of Approximate Reasoning* 55(9):1866–1889.

Diller, M.; Keshavarzi Zafarghandi, A.; Linsbichler, T.; and Woltran, S. 2020. Investigating subclasses of abstract dialectical frameworks. *Argument & Computation* 11(1-2):191–219.

Dubois, D., and Prade, H. 1986. The principle of minimum specificity as a basis for evidential reasoning. In *International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 75–84.

Dubois, D., and Prade, H. 1993. Fuzzy sets and probability: misunderstandings, bridges and gaps. In *[Proceedings 1993] Second IEEE International Conference on Fuzzy Systems*, 1059–1068. IEEE.

Dubois, D., and Prade, H. 1998. Possibility theory: qualitative and quantitative aspects. In *Quantified representation of uncertainty and imprecision*. Springer. 169–226.

Dung, P. M. 1995. On the Acceptability of Arguments and its Fundamental Role in Nonmonotonic Reasoning, Logic Programming and n-Person Games. *Artificial Intelligence* 77(2):321–358.

Gaggl, S. A., and Strass, H. 2014. Decomposing abstract dialectical frameworks. In *Computational Models of Argument: Proceedings of COMMA*, 281–292.

Heyninck, J., and Kern-Isberner, G. 2020. An epistemic interpretation of abstract dialectical argumentation. In *Computational Models of Argument: Proceedings of COMMA 2020*, 227 – 238.

Hunter, A., and Thimm, M. 2017. Probabilistic reasoning with abstract argumentation frameworks. *Journal of Artificial Intelligence Research* 59:565–611.

Janssen, J.; De Cock, M.; and Vermeir, D. 2008. Fuzzy argumentation frameworks. In *Information processing and management of uncertainty in knowledge-based systems*, 513–520.

Kern-Isberner, G.; Beierle, C.; and Brewka, G. 2020. Syntax splitting= relevance+ independence: New postulates for nonmonotonic reasoning from conditional belief bases. In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning*, 560–571.

Kleene, S. C.; De Bruijn, N.; de Groot, J.; and Zaanen, A. C. 1952. *Introduction to metamathematics*, volume 483. van Nostrand New York.

Lifschitz, V.; Pearce, D.; and Valverde, A. 2001. Strongly equivalent logic programs. *ACM Transactions on Computational Logic (TOCL)* 2(4):526–541.

Linsbichler, T. 2014. Splitting abstract dialectical frameworks. In *Computational Models of Argument: Proceedings of COMMA*, 357–368.

Polberg, S.; Wallner, J. P.; and Woltran, S. 2013. Admissibility in the abstract dialectical framework. In *International Workshop on Computational Logic in Multi-Agent Systems*, 102–118. Springer.

Sette, A. M. 1973. On the propositional calculus p1. *Mathematica Japonicae*.

Wu, J.; Li, H.; Oren, N.; and Norman, T. J. 2016. Gödel fuzzy argumentation frameworks. In *Computational Models of Argument: Proceedings of COMMA*, 447–458.