Ordinal Conditional Functions for Abstract Argumentation

Kenneth SKIBA and Matthias THIMM
Artificial Intelligence Group, University of Hagen, Germany

Abstract. We interpret and formalise ordinal conditional functions (OCFs) in abstract argumentation frameworks based on ideas and concepts defined for conditional logics. There, these functions are used to rank interpretations, and we adapt them to rank extensions instead. Using conflict-freeness and admissibility as two essential principles to define the semantics of OCFs, we obtain a framework that allows to rank sets of arguments wrt. their plausibility. We analyse the properties of this framework in-depth, and in doing so we establish a formal bridge between the approaches of abstract argumentation and conditional logics.

Keywords. Abstract Argumentation, Ranking Functions

1. Introduction

Abstract argumentation frameworks (AF) [1] have gathered research interest as a model for rational decision-making in presence of conflicting information. Using AFs, arguments and attacks can be represented as nodes and edges, respectively, of a directed graph. In order to reason over AFs extension semantics were defined, which are functions such that a set of arguments can be considered jointly acceptable. Recently Skiba et al. [2] generalised this reasoning process to rank sets of arguments based on their plausibility. Another used reasoning formalism is conditional logic, which studies conditionals like “if A then B” written as (B | A). So given the information that A is true it is more “believable” that B is true, than B being not true. In order to define a value of believability, ordinal conditional functions (OCF) (also known as ranking functions) were defined [3]. These functions can be used to rank possible worlds according to their plausibility. One example of an OCF is the System Z ranking function [4], which exhibits particularly good reasoning properties.

In recent years, the relationship between argumentation and conditional logic was investigated in [5,6,7,8] and by Weydert [9,10]. While abstract argumentation usually only provides a criterion to determine whether a set of arguments is jointly accepted or not, OCFs on the other hand can rank possible worlds according to their plausibility. In this paper, we want to use these ideas from conditional logic to reason in abstract argumentation. Our goal is to rank sets of arguments according to their plausibility, i.e., we can state not only whether a set of arguments is accepted or not, but also state that a set is more plausible than another one. In particular, we can rank sets of arguments, which are not jointly acceptable w.r.t. extension semantics, for example, we can say that out of two conflicting sets one of them is more plausible. One potential application of such a
ranking is decision-making in presence of constraints, where a solution (represented as a set of arguments) satisfies constraints that cannot be satisfied by a set of arguments under extension semantics. One possible way to still make a decision would be to select the most plausible sets of arguments, which are satisfying the constraints.

To achieve such a ranking of sets of arguments, we will use the guiding principles of admissible reasoning for abstract argumentation frameworks namely conflict-freeness and admissibility to develop ordinal conditional functions for abstract argumentation. In order to link

We use the attack relation between two arguments as a bridge between abstract argumentation and conditional logics. Since there can be a number of functions satisfying the defined principles, we develop a model-based reasoning technique inspired by System Z. This System Z ranking function allows us to model plausibility values for each set of arguments $I$ being in while a different set $O$ is out. These plausibility values can be used to rank sets of arguments, and therefore continue recent work about extension-ranking semantics started in [2].

This paper is structured as follows. We recall all necessary preliminaries about abstract argumentation and conditional logics in Section 2. Section 3 introduces OCFs for abstract argumentation. In Section 4, we look at OCFs based on System Z. A extension-ranking semantics is introduced in Section 5 as well as an in-depth investigation of the properties of that semantics is presented. We conclude this paper in Section 6 with a discussion about related work.

2. Preliminaries

In this section, we recall all necessary definitions of abstract argumentation and conditional logics.

2.1. Abstract Argumentation

Argumentation frameworks [1] are a formalism that allows the representation of conflicts between pieces of information using arguments and attacks between arguments.

**Definition 1.** An abstract argumentation framework (AF) is a directed graph $AF = (A, R)$ where $A$ is a finite set of arguments and $R$ is an attack relation $R \subseteq A \times A$.

An argument $a$ is said to attack an argument $b$ if $(a, b) \in R$. We say that an argument $a$ is defended by a set $S \subseteq A$ if every argument $b \in A$ that attacks $a$ is attacked by some $c \in S$. For $a \in A$ define $a^- = \{b \mid (b, a) \in R\}$ and $a^+ = \{b \mid (a, b) \in R\}$, so the set of attackers of $a$ and the set of arguments attacked by $a$. For a set of arguments $S \subseteq A$ we extend these definitions to $S^+ = \bigcup_{a \in S} a^+$ and $S^- = \bigcup_{a \in S} a^-$, respectively. For two graphs $AF = (A, R)$ and $AF' = (A', R')$, we define $AF \cup AF' = (A \cup A', R \cup R')$.

To reason with abstract argumentation frameworks a number of different semantical notions have been developed, like the extension-based or the labelling-based approaches, for an overview see [11]. Both these approaches are handling sets of arguments, which can be considered jointly acceptable. The extension-based semantics are relying on two basic concepts: conflict-freeness and defence.

**Definition 2.** Given $AF = (A, R)$, a set $E \subseteq A$ is:
skeptically accepted, we can define the status of any (set of) argument(s), namely

Consider the argumentation framework

Example 1. note these sets of arguments. \( \sigma \) and \( \text{rejected} \) are denoted (respectively) co\( \sigma \)-extension, a ground complete extension; the unique grounded extension (gr) iff it is the \( \subseteq \)-minimal complete extension; a stable extension (st) iff \( E^+ = A \setminus E \).

The sets of extensions of an argumentation framework \( \text{AF} \), for these four semantics, are denoted (respectively) co\( \text{AF} \), pr\( \text{AF} \), gr\( \text{AF} \) and st\( \text{AF} \). Based on these semantics, we can define the status of any (set of) argument(s), namely skeptically accepted (belonging to each \( \sigma \)-extension), credulously accepted (belonging to some \( \sigma \)-extension) and rejected (belonging to no \( \sigma \)-extension). Given an argumentation framework \( \text{AF} \) and an extension semantics \( \sigma \), we use (respectively) sk\( \sigma \)\( \text{AF} \), cr\( \sigma \)\( \text{AF} \) and rej\( \sigma \)\( \text{AF} \) to denote these sets of arguments.

Example 1. Consider the argumentation framework \( \text{AF} = (A, R) \) depicted as a directed graph in Figure 1, with the nodes corresponding to the arguments \( A = \{a, b, c, d\} \), and the edges corresponding to the attacks \( R = \{(a, b), (b, c), (c, d), (d, c)\} \). The sets \{a\}, \{a, c\} and \{a, d\} are complete extensions of \( \text{AF} \), while only \{a, c\} and \{a, d\} are stable.

For more details about these semantics (and other ones defined in the literature), we refer the interested reader to [1,11].

2.2. Conditional Logics

In order to define the usual propositional language \( \mathcal{L}(A_t) \) over \( A_t \) we use a set of atoms \( A_t \) and connectives \( \land \) (and), \( \lor \) (or) and \( \neg \) (negation). The function \( \omega : A_t \to \{T, F\} \) defines an interpretation (or possible world) \( \omega \) for \( \mathcal{L}(A_t) \). \( \Omega(A_t) \) denotes the set of all interpretations. An interpretation \( \omega \) satisfies an atom \( a \in A_t \) (\( \omega \models a \)), iff \( \omega(a) = T \). As a conditional we consider structures like \( (\psi \mid \phi) \), which can be read as “if \( \phi \) then (usually) \( \psi \)”. Informally speaking, an interpretation \( \omega \) verifies a conditional \( (\psi \mid \phi) \) iff it satisfies both antecedent (\( \phi \)) and conclusion (\( \psi \)) \( ((\psi \mid \phi) (\omega) = 1) \); it falsifies if it satisfies the antecedent but not the conclusion \( ((\psi \mid \phi) (\omega) = 0) \); otherwise the conditional is not applicable. A conditional is satisfied by \( \omega \) if it does not falsify it.

We use ordinal conditional functions (OCFs) (also called ranking functions) [3], \( \kappa : \Omega(A_t) \to \mathbb{N} \cup \{\infty\} \) to denote a plausibility degree of interpretations and define \( \kappa(\phi) := \min\{\kappa(\omega) \mid \omega \models \phi \} \). An OCF \( \kappa \) satisfies a set \( \Delta \) of conditionals, if for each \( (\psi \mid \phi) \in \Delta \), \( \kappa(\phi \land \psi) < \kappa(\phi \land \neg \psi), \) i.e., verifying a conditional is more plausible than falsifying it. Since the set of all satisfying OCFs may be difficult to handle, one usually relies
on model-based inference for reasoning. In this paper, we will focus on the System Z ranking function [4] as an example for model-based inference.

**Definition 4.** A conditional \((\psi \mid \phi)\) is tolerated by a finite set of conditionals \(\Delta\) if there is a possible world \(\omega\), which verifies \((\psi \mid \phi)\) and does not falsify any conditional \((\psi' \mid \phi')\) \(\in \Delta\). The Z-Partitioning \((\Delta_0, \ldots, \Delta_n)\) of \(\Delta\) is defined as:

- \(\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}\)
- \(\Delta_1, \ldots, \Delta_n\) is the Z-Partitioning of \(\Delta \setminus \Delta_0\)

For \(\delta \in \Delta : Z_\Delta(\delta) = i\) iff \(\delta \in \Delta_i\) and \((\delta_0, \ldots, \delta_n)\) is the Z-Partitioning of \(\Delta\). We define a ranking function \(\kappa^Z\) via \(\kappa^Z_\omega(\omega) = \max\{Z_\Delta(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1\), with \(\max \emptyset = -1\).

**Example 2 ([4]).** Consider the following set of conditionals \(\Delta\) about the flying ability of penguins.

\(\delta_1:\) “birds fly” \((f \mid b)\) \(\delta_2:\) “penguins are birds” \((b \mid p)\)

The Z-Partitioning of \(\Delta\) is \(\Delta_0 = \{\delta_1\}\) and \(\Delta_1 = \{\delta_2, \delta_3\}\), because \(\Delta_0\) can be tolerated by all conditionals, while \(\delta_2\) and \(\delta_3\) cannot be tolerated by \(\Delta\). We can calculate the plausibility value of interpretations \(\omega\). For example, a flying penguin \((\omega(p) = \omega(b) = \omega(f) = T)\) receives a value of \(\kappa^Z_\omega(\omega) = 1\).

### 3. Ordinal Conditional Functions for Abstract Argumentation

In this section we define OCFs for abstract argumentation. We define a function \(\kappa\) with two parameters (\(I\) and \(O\)) to calculate a numerical plausibility value. These parameter are sets of arguments where the first set \(I\) is considered in, and the second set is considered out. So \(\kappa(I, O) = 0\) means that the set \(I\) being in and the set \(O\) being out is not surprising. Note that our OCF need two parameters instead of only one, like in conditional logics, since abstract argumentation is missing the notion of negation. Hence, the second parameter is used to model a negation.

**Definition 5.** Let \(AF = (A, R)\) be an AF. A OCF for \(AF\) is a function \(\kappa : 2^A \rightarrow \mathbb{N} \cup \{\infty\}\) with \(\kappa^{-1}(0) \neq \emptyset\).

For sets \(I, O \subseteq A\) we abbreviate

\[
\kappa(I, O) = \min\{\kappa(S) \mid I \subseteq S, S \cap O = \emptyset\}
\]

\[
\kappa(I, O) = \infty \text{ if } I \cap O \neq \emptyset
\]

**Example 3.** Consider \(AF_1 = (\{a, b\}, \{(a, b)\})\). One exemplary OCF \(\kappa^C(I, O)\) returns the number of conflicts in \(I\), i.e., for \(\{a, b\}\) \(\kappa^C(\{a, b\}, \emptyset) = 1\). For any other set \(S \subset \{a, b\}\) like \(S = \{a\}\) we have \(\kappa^C(S, \emptyset) = \kappa^C(S, \{b\}) = 0\) and \(\kappa^C(S, S) = \infty\).

The following definitions are inspired by OCFs in conditional logic. However, while conditional logic semantics follow a single principle regarding conditional acceptance (“a conditional is accepted if its verification is more plausible than its violation”), for admissible reasoning in abstract argumentation we have two guiding principles:
The first principle is also called conflict-freeness, i.e., a set does not contain two arguments, which share an attack. So conflicting sets are less plausible than conflict-free sets. The second principle is admissibility, so a set, which defends itself from all possible attackers, is at least as plausible as set not defending itself. Implementing these two principles for OCFs gives us:

**Definition 6.** Let \( AF = (A, R) \) be an AF and \( a, b \in A \).

- An OCF \( \kappa \) accepts an attack \( (a, b) \) with \( a \neq b \) if \( \kappa(\{a\}, \{b\}) < \kappa(\{a, b\}, \emptyset) \).
- An OCF \( \kappa \) possibly reinstates an argument \( a \in A \) if \( \kappa(S \cup \{a\}, a^-) \leq \kappa(S, \{a\} \cup a^-) \) for all \( S \subseteq A \) with \( S \cap (a^- \cup a^+) = \emptyset \).

Intuitively, for an OCF to accept an attack \( (a, b) \) means that it is more plausible that argument \( a \) is in and \( b \) is out, than both \( a \) and \( b \) being in at the same time. For an OCF to possibly reinstate an argument \( a \) means that if all attackers of \( a \) are out, then it is at least as plausible that \( a \) is in than out.

Next we want to denote when an AF is satisfied by an OCF, i.e., when we can define an OCF satisfying all principles defined above for an AF.

**Definition 7.** An OCF \( \kappa \) satisfies an argumentation framework \( AF = (A, R) \) if it accepts all attacks in \( R \) and possibly reinstates all arguments in \( A \).

**Example 4.** Consider \( AF_2 = (\{a, b\}, \{(a, b)\}) \). So the following statements have to hold:

1. \( \kappa(\{a\}, \{b\}) < \kappa(\{a, b\}, \emptyset) \)
2. \( \kappa(\{a\}, \emptyset) \leq \kappa(\emptyset, \{a\}) \)
3. \( \kappa(\{b\}, \{a\}) \leq \kappa(\emptyset, \{a, b\}) \)

Table 1 depicts an OCF that satisfies \( AF_2 \).

Note that if an AF contains a self-attacking argument \( a \), then there is no OCF that satisfies it. Because to accept attack \( (a, a) \) it has to hold that \( \kappa(\{a\}, \{a\}) < \kappa(\{a\}, \emptyset) \), which is impossible, since \( \kappa(\{a\}, \{a\}) = \infty \).

### 4. The System Z Ranking Function for Abstract Argumentation

In this section, we want to define an OCF inspired by System Z. The basic idea of System Z is that a conditional \( (B \mid A) \) is tolerated by a set of conditionals, if it is confirmed by
conflict-free and every argument contained in $S$ does not contain any attackers of cause, for an attack admissibility of argument $c$.

Definition 9. Let $AF = (A, R)$ be an argumentation framework.

- A set $S \subseteq A$ verifies an attack $(a, b)$ iff $a \in S$ and $b \notin S$.
- A set $S \subseteq A$ violates an attack $(a, b)$ iff $a \in S$ and $b \in S$.
- A set $S \subseteq A$ satisfies an attack $(a, b)$ iff it does not violate it.

Intuitively speaking, a set satisfies an attack if this set does not contain any two conflicting arguments. So for an AF $AF_3 = \{(a, b, c), ((a, b), (b, c))\}$, we can observe, that the set $S_1 = \{a\}$ verifies the attack $(a, b)$ and does not violate the attack $(b, c)$, while the set $S_2 = \{a, b\}$ verifies the attack $(b, c)$, however $S_2$ violates attack $(a, b)$.

To satisfy attack admissibility of an argument, we know that, if all the attackers of the argument are out, then the argument itself should be in.

Definition 10. Let $AF = (A, R)$ be an argumentation framework.

- A set $S \subseteq A$ verifies attack admissibility of $a \in A$ iff $a \not\in S$ and $b \notin S$ for all $b \in a^-$. A set $S \subseteq A$ violates attack admissibility of $a \in A$ iff $a \notin S$ and $b \notin S$ for all $b \in a^-$. A set $S \subseteq A$ satisfies attack admissibility of $a \in A$ iff it does not violate it.

We recall $AF_3 = \{(a, b, c), ((a, b), (b, c))\}$, we see that the set $S_3 = \{a, c\}$ verifies attack admissibility of argument $c$, because the only attacker of $c$, $b$, is not part of $S_3$ and one of $b$'s attackers is contained in $S_3$.

Now we combine both these definitions and define when an attack can be tolerated.

Definition 11. Let $AF = (A, R)$ be an argumentation framework. A set $P \subseteq R$ tolerates an attack $(a, b)$ iff there is a set $S \subseteq A$ that

1. verifies $(a, b)$,
2. satisfies each attack in $P$, and
3. satisfies attack admissibility of each $c \in A$

So in order to tolerate an attack, we need to find a set of arguments $S$ s.t. $S$ is conflict-free and every argument contained in $S$ has to be defended. Recall $AF_3 = \{(a, b, c), ((a, b), (b, c))\}$, then the attack $(b, c)$ is not tolerated by $\{(a, b), (b, c)\}$. Because, for $(b, c)$ to be verified for any set $S$, it has to hold that $b \in S$. Then to not violate $(a, b)$ $a$ is not allowed to be contained in $S$. However, then we have the problem that $S$ does not contain any attackers of $a$, meaning that attack admissibility of $a$ is violated.

With these definitions, we can define the OCF $\kappa^Z$ for an AF $AF$. 

Skiba and Thimm / Ordinal Conditional Functions for Abstract Argumentation
Definition 11. Let \( \mathcal{AF} = (\mathcal{A}, \mathcal{R}) \) be an argumentation framework. Then the Z-attack-Partitioning \((\mathcal{R}_0, \ldots, \mathcal{R}_n)\) with \( \mathcal{R}_0 \cup \cdots \cup \mathcal{R}_n \subseteq \mathcal{R} \) is defined as:

- \( \mathcal{R}_0 = \{ r \in \mathcal{R} \mid \mathcal{R} \text{ tolerates } r \} \)
- \( (\mathcal{R}_1, \ldots, \mathcal{R}_n) \) is the Z-attack-Partitioning of \( \mathcal{R} \setminus \mathcal{R}_0 \)

For \( r \in \mathcal{R} \) define \( Z_{\mathcal{R}}(r) = i \) if \( r \in \mathcal{R}_i \) and

\[
\kappa^Z(S, X) = \max\{Z(r) \mid S \text{ violates } r\} + 1
\]

where \( X \subseteq \mathcal{A} \) is any set s.t. \( S \cap X = \emptyset \).

So all attacks in \( \mathcal{R}_0 \) are tolerated by the set of attacks of \( \mathcal{AF} \), while attacks in \( \mathcal{R}_1 \) are only tolerated if we remove all attacks from \( \mathcal{R}_0 \). Now we can state when a set of arguments is more plausible than another one, i.e., if the first set violates either no attacks or only attacks which are in lower levels. In a sense these levels represent the impact of each attack in an \( \mathcal{AF} \). Hence, it is more important to satisfy a single highly ranked attack than to satisfy multiple lowly ranked attacks.

Example 5. Consider Example 1 again. The Z-attack-Partitioning of \( \mathcal{R} \) is \((\mathcal{R}_0, \mathcal{R}_1)\) with

\[
\mathcal{R}_0 = \{(a, b), (c, d), (d, c)\}
\]
\[
\mathcal{R}_1 = \{(b, c)\}
\]

Table 2 depicts \( \kappa^Z_{\mathcal{AF}} \) for \( \mathcal{AF} \) from Example 1.

Next, we want to prove, that the function \( \kappa^Z \) satisfies an \( \mathcal{AF} \) if \( \kappa^Z \) is defined.

Theorem 1. If \( \kappa^Z \) is defined, then \( \kappa^Z \) satisfies \( \mathcal{AF} \).

Proof. Let \( \mathcal{AF} = (\mathcal{A}, \mathcal{R}) \) be an \( \mathcal{AF} \). In order to show that \( \kappa^Z \) satisfies \( \mathcal{AF} \), we need to prove, that \( \kappa^Z \) satisfies both principles of an OCF, i.e., acceptance of attacks and possibly reinstating an argument.

Case: accept attack. Let \( (a, b) \in \mathcal{R} \) with \( a \neq b \) an attack, it has to hold that \( \kappa^Z(\{a\}, \{b\}) < \kappa^Z(\{a, b\}, \emptyset) \). We know that \( \kappa^Z(\{a\}, \{b\}) = 0 \), because \( \{a\} \) can only violate the attack \( (a, a) \), which can not exist. Hence, it is enough to show, that \( \kappa^Z(\{a, b\}, \emptyset) > 0 \). Since, \( (a, b) \) exists, we know that \( \{a, b\} \) violates this attack, and therefore \( \kappa^Z(\{a, b\}, \emptyset) > 0 \).

Case: argument possibly reinstated. Let \( a \in \mathcal{A} \) be an argument. Assume \( \kappa^Z(\mathcal{S} \cup a, a^-) > \kappa^Z(\mathcal{S}, a \cup a^-) \) for some \( \mathcal{S} \subseteq \mathcal{A} \) with \( \mathcal{S} \cap (a^- \cup a^+) = \emptyset \). This is only possible, if \( \mathcal{S} \cup \{a\} \) violates an attack \( r \in \mathcal{R} \) and \( \mathcal{S} \) does not violate \( r \). So, there is one argument \( b \in \mathcal{S} \) s.t. \( r = (a, b) \) or \( r = (b, a) \) and \( a \neq b \). Hence, \( b \in a^- \cup a^+ \). However, because of \( \mathcal{S} \cap (a^- \cup a^+) = \emptyset \) we know that, there can not be such an argument \( b \in \mathcal{S} \) with \( b \in a^- \cup a^+ \). Therefore \( \kappa^Z(\mathcal{S} \cup a, a^-) > \kappa^Z(\mathcal{S}, a \cup a^-) \) is impossible.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \kappa^{-1}(i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(({b, c}, X), ({a, b, c}, X), ({b, c, d}, X), ({a, b, c, d}, X))</td>
</tr>
<tr>
<td>1</td>
<td>(({a, b}, X), ({c, d}, X), ({a, b, d}, X), ({a, c, d}, X))</td>
</tr>
<tr>
<td>0</td>
<td>((\emptyset, X), ({a}, X), ({b}, X), ({c}, X), ({d}, X), ({a, c}, X), ({a, b}, X), ({a, d}, X), ({a}, X))</td>
</tr>
</tbody>
</table>

Table 2. The OCF \( \kappa^Z \), where for every pair \((I, X) \subseteq A\) is any set s.t. \( I \cap X = \emptyset \).
Besides being undefined for AF with self-attacks, $\kappa^Z$ is also undefined for AFs without a stable extension. Let $\text{AF}_4 = (\{a,b,c\},\{(a,b),(b,c),(c,a)\})$ be an AF. If we try to tolerate $(a,b)$ by $\{(a,b),(b,c),(c,a)\}$, then we know that, we need to verify $(a,b)$ so $a \in S$. However, this also means that $b,c \notin S$, which entails that attack admissibility of $c$ is violated. Similar we can show, that $(b,c)$ and $(c,a)$ cannot be tolerated either. So, we cannot define a Z-attack-Partitioning for $\text{AF}_4$. Next, we show that in general it holds that if an AF does not have a stable extension, then $\kappa^Z$ is undefined.

Theorem 2. $\kappa^Z$ is undefined for AF if $\text{st}(\text{AF}) = \emptyset$.

Proof. Let $\text{AF} = (A,R)$ be an AF. We will show the contraposition, so if $\kappa^Z$ is defined for AF, then $\text{st}(\text{AF}) \neq \emptyset$. Let $\kappa^Z$ be defined. So we can find a Z-attack-Partitioning $(R_0,...,R_n)$. For every attack $r$ in $R_0$ we know that there is a set $S$ s.t. $r$ is verified, every attack is satisfied and attack admissibility of every argument $a \in A$ is satisfied. We show that $S$ is stable. First, $S$ has to be conflict-free, otherwise there is an attack, which is violated. Next we show that $S \cup S^+ = A$, so we need to show, that every argument not in $S$ is attacked by $S$. Let $b \notin S$, then because attack admissibility of $b$ is satisfied we know that there is an argument $c \in b^−$ with $c \in S$, hence we have found an attacker of $b$ which is part of $S$.

Looking at the levels of a Z-attack-Partitioning in detail, we observe, that if an attack $(a,b)$ is in $R_0$, then $a$ is credulously admissible accepted in AF.

Theorem 3. For any AF $\text{AF} = (A,R)$ and Z-attack-Partitioning $(R_0,...,R_n)$. If $(a,b) \in R_0$, then $a$ is credulously accepted w.r.t. admissible semantics.

Proof. Let $\text{AF} = (A,R)$ be an AF, $(R_0,...,R_n)$ a Z-attack-Partitioning of $R$ and $(a,b) \in R_0$, then $(a,b)$ is tolerated by $R$, meaning that there is an $S \subseteq A$ s.t. $(a,b)$ is verified, each attack in $R$ is satisfied by $S$ and attack admissibility of each argument $c \in A$ is satisfied. In order to verify $(a,b)$, we know that $a \in S$ and $b \notin S$. Also it has to hold for all $c \in a^{−}$ that $c \notin S$. So all attackers of $a$ are out. Next, we will show that $S$ is admissible. $S$ is conflict-free, otherwise, one attack would be violated. We know for every $d \in S$ that no attacker of $d$ is in $S$. In order to not violate attack admissibility, we know for every $e \notin S$ that at least one attacker of $e$ has to be in $S$, meaning that $S$ attacks every argument not contained in $S$. Hence, for every attacker $b$ of an argument $a \in S$ we have an argument $d \in S$ s.t. $d$ attacks $b$. So $S$ is admissible, and therefore $a$ is part of some admissible extension of $\text{AF}$ making $a$ credulously accepted w.r.t. admissible semantics.

5. Extension-ranking Semantics based on System Z

First, we recall the definitions from [2] for extension-ranking semantics.

5.1. Extension-Ranking Semantics

Extension-ranking semantics defined in [2] are a generalisation of extension-based semantics. Using them, we can state not only that a set of arguments is jointly accepted or not, but also we can say whether a set $E_1$ is more plausible than a set $E_2$. 
Definition 12. An extension ranking on \( \text{AF} \) is a preorder\(^1\) over the powerset of arguments \( 2^\mathcal{A} \). An extension-ranking semantics \( \tau \) is a function that maps each \( \text{AF} \) to an extension ranking \( \succeq^\tau_{\text{AF}} \) on \( \text{AF} \).

For an extension-ranking semantics \( \tau \), an extension ranking \( \succeq^\tau_{\text{AF}}, E, E' \subseteq A \), and for \( E \succeq^\tau_{\text{AF}} E' \) we say that \( E \) is at least as plausible as \( E' \) by \( \tau \) in \( \text{AF} \).

Using the OCF \( \kappa^Z \), we can define an extension-ranking semantics. So we can state that a set of arguments \( E \) is more plausible than another one \( E' \), if the OCF \( \kappa^Z \) returns a lower value for \( E \) than for \( E' \).

Definition 13. Let \( \text{AF} = (A, R) \) be an \( \text{AF} \) and \( E, E' \subseteq A \). Define the System Z extension-ranking semantics \( \succeq^Z_{\text{AF}} \) via

\[
E \succeq^Z_{\text{AF}} E' \text{ iff } \kappa^Z(E, A \setminus E) \leq \kappa^Z(E', A \setminus E')
\]

So \( E \) is at least as plausible as \( E' \), if \( E \) being considered in and all arguments not in \( E \) being out, is more plausible than \( E' \) being considered in and all arguments not in \( E' \) being out.

Example 6. Consider again \( \text{AF} \) from Example 1. Then Table 3 depicts the ranking corresponding to \( \succeq^Z_{\text{AF}} \). We see, that all conflict-free sets are part of the most plausible sets, while sets with conflicts are ranked worse. Also, the number of conflicts is not as important as for the approaches of [2]. In their approaches, it always holds that \( \{b, c\} \) is ranked strictly better than \( \{b, c, d\} \). While for \( \kappa^Z \) these two sets are ranked equally.

5.2. Study of the System Z Extension-ranking Semantics

Next, we want to evaluate \( \succeq^Z_{\text{AF}} \) based on principles defined by [2].

We begin with \( \sigma \)-generalisation, which states that sets of arguments, which satisfies extension semantics \( \sigma \) should also be ranked best by an extension-ranking semantics and every set not satisfying \( \sigma \) should be ranked worse. In Example 6, we can see that \( \succeq^Z_{\text{AF}} \) violates \( \sigma \)-generalisation for \( \sigma \in \{\text{ad,co,pr,gr,si}\} \), because the set \( \{b, d\} \) is not admissible, however, it is ranked as a most plausible set. Therefore \( \succeq^Z_{\text{AF}} \) cannot satisfy \( \sigma \)-generalisation for any admissible based semantics \( \sigma \).

The next properties (composition and decomposition) states that unconnected arguments should not influence a ranking.

Theorem 4. \( \succeq^Z_{\text{AF}} \) satisfies composition. Where \( \tau \) satisfies composition if for every \( \text{AF} \) such that \( \text{AF} = (A_1, R_1) \cup (A_2, R_2) \) and \( E, E' \subseteq A_1 \cup A_2 \):

\( A preorder is a (binary) relation that is reflexive (E \subseteq E for all E) and transitive (E_1 \subseteq E_2 and E_2 \subseteq E_3 implies E_1 \subseteq E_3) \)

Table 3. The ranking for \( \text{AF} \) based on \( \succeq^Z_{\text{AF}} \).

\[
\begin{array}{ccccccc}
\emptyset \succeq^Z_{\text{AF}} \{a\} & \succeq^Z_{\text{AF}} \{b\} & \succeq^Z_{\text{AF}} \{c\} & \succeq^Z_{\text{AF}} \{d\} & \succeq^Z_{\text{AF}} \{a,c\} & \succeq^Z_{\text{AF}} \{b,d\} & \succeq^Z_{\text{AF}} \{a,d\} \\
\succeq^Z_{\text{AF}} \{a,b\} & \succeq^Z_{\text{AF}} \{c,d\} & \succeq^Z_{\text{AF}} \{a,b,d\} & \succeq^Z_{\text{AF}} \{a,c,d\} \\
\succeq^Z_{\text{AF}} \{b,c\} & \succeq^Z_{\text{AF}} \{a,b,c\} & \succeq^Z_{\text{AF}} \{b,c,d\} & \succeq^Z_{\text{AF}} \{a,b,c,d\}
\end{array}
\]
if for every \((\text{Weak Decomposition})\) \(\kappa\) en if \(\kappa\) Let \(\kappa\)inition we know that \(\kappa\) Similar holds for \(\kappa\).

Proof. Let \(AF = (A_1, R_1) \cup (A_2, R_2)\) be an AF and \(E, E' \subseteq A_1 \cup A_2\). For composition we need to show that, if \(\kappa^2(E \cap A_1, A_1 \setminus E) \leq \kappa^2(E' \cap A_1, A_1 \setminus E')\) and \(\kappa^2(E \cap A_2, A_2 \setminus E) \leq \kappa^2(E' \cap A_2, A_2 \setminus E')\) then \(\kappa^2(E, A \setminus E) \leq \kappa^2(E', A \setminus E')\). By definition of \(\kappa^2\) we know that \(\kappa^2(E, A \setminus E)\) is the maximal value between \(\kappa^2(E \cap A_1, A_1 \setminus E)\) and \(\kappa^2(E \cap A_2, A_2 \setminus E)\), because if an attack \(r_1\) is violated by \(E\), then \(r_1\) is also violated by either \(E \cap A_1\) or \(E \cap A_2\). Similar holds for \(\kappa^2(E', A \setminus E')\). So we have to check four possible cases for \(\max(\kappa^2(\cap A_1, A_1 \setminus E), \kappa^2(E \cap A_2, A_2 \setminus E)) \leq \max(\kappa^2(\cap A_1, A_1 \setminus E'), \kappa^2(E' \cap A_2, A_2 \setminus E'))\).

1. \(\kappa^2(E \cap A_1, A_1 \setminus E) \leq \kappa^2(E' \cap A_1, A_1 \setminus E')\)
2. \(\kappa^2(E \cap A_2, A_2 \setminus E) \leq \kappa^2(E' \cap A_2, A_2 \setminus E')\)
3. \(\kappa^2(E \cap A_1, A_1 \setminus E) \leq \kappa^2(E' \cap A_2, A_2 \setminus E')\)
4. \(\kappa^2(E \cap A_2, A_2 \setminus E) \leq \kappa^2(E' \cap A_1, A_1 \setminus E')\)

Case 1 and 2 are clear via definition. For case 3 we know that \(\kappa^2(E \cap A_1, A_1 \setminus E) \geq \kappa^2(E \cap A_2, A_2 \setminus E)\) and \(\kappa^2(E \cap A_1, A_1 \setminus E') \leq \kappa^2(E' \cap A_2, A_2 \setminus E')\), but we also know that \(\kappa^2(E \cap A_1, A_1 \setminus E) \leq \kappa^2(E' \cap A_1, A_1 \setminus E')\), which proves case 3. Case 4 is similar to case 3.

For decomposition, we see that \(\preceq^\kappa_{AF}\) violates it. Recall that \(\tau\) satisfies decomposition if for every AF such that \(AF = (A_1, R_1) \cup (A_2, R_2)\) and \(E, E' \subseteq A_1 \cup A_2\):

if \(E \preceq^\kappa_{AF} E'\) then \(\left\{ \begin{align*}
E \cap A_1 & \preceq^\tau_{AF_1} E' \cap A_1 \\
E \cap A_2 & \preceq^\tau_{AF_2} E' \cap A_2
\end{align*} \right\}\)

Example 7. Let \(AF_2 = \{(a, b, c, d, e), (a, b, c, (d, e))\}\) be an AF. This AF can be split into two disjoint AFs \(AF_{5,1} = \{(a, b, c), (a, b, (c, d))\}\) and \(AF_{5,2} = \{(d, e), (d, (e))\}\). The Z-attack-Partitioning of \(R_2\) is \(R_{5,0} = \{(a, b), (d, e)\}\) and \(R_{5,1} = \{(b, c)\}\). Let \(E = \{(a, b, d, e)\} \) and \(E' = \{(b, c, d)\} \), then \(\kappa^2(E, A_3 \setminus E) = 1\) and \(\kappa^2(E', A_3 \setminus E') = 2\). However, we have \(\kappa^2(E \cap A_5, A_5 \setminus E) = 1\) and \(\kappa^2(E' \cap A_5, A_5 \setminus E') = 0\). This shows, that decomposition is violated.

Decomposition is violated, because \(\preceq^\kappa_{AF}\) focuses on a global view. Violating \((b, c)\) is worse, than violating any other attack. However, \(\preceq^\kappa_{AF}\) satisfies a weak version of decomposition, where instead of satisfying \(E \cap A_1 \preceq^\tau_{AF_1} E' \cap A_1\) for both disjoint AFs, it is enough if \(\kappa^2\) satisfy this for one AF.

Definition 14 (Weak Decomposition). Let \(\tau\) be an extension-ranking semantics. \(\tau\) satisfies weak decomposition if for every AF such that \(AF = (A_1, R_1) \cup (A_2, R_2)\) and \(E, E' \subseteq A_1 \cup A_2\): if \(E \preceq^\kappa_{AF} E'\) then \(E \cap A_1 \preceq^\tau_{AF_1} E' \cap A_1\) or \(E \cap A_2 \preceq^\tau_{AF_2} E' \cap A_2\).

Theorem 5. \(\preceq^\kappa_{AF}\) satisfies weak decomposition.

Proof. Let \(AF = (A_1, R_1) \cup (A_2, R_2)\) be an AF and \(E, E' \subseteq A_1 \cup A_2\). In order to prove weak decomposition we have to show, that if \(\kappa^2(E, A \setminus E) \leq \kappa^2(E', A \setminus E')\) then \(\kappa^2(E \cap A_1, A_1 \setminus E) \leq \kappa^2(E', A_1 \setminus E')\) or \(\kappa^2(E \cap A_2, A_2 \setminus E) \leq \kappa^2(E', A_2 \setminus E')\). By definition we know that \(\kappa^2(E, A \setminus E) = \max(\kappa^2(E \cap A_1, A_1 \setminus E), \kappa^2(E \cap A_2, A_2 \setminus E))\) and
similar for $E'$. So, we have $\max(\kappa^z(E \cap A_1, A_1 \setminus E), \kappa^z(E \cap A_2, A_2 \setminus E)) \leq \max(\kappa^z(E' \cap A_1, A_1 \setminus E'), \kappa^z(E' \cap A_2, A_2 \setminus E'))$. Hence, we have four cases to check.

1. $\kappa^z(E \cap A_1, A_1 \setminus E) \leq \kappa^z(E' \cap A_1, A_1 \setminus E')$
2. $\kappa^z(E \cap A_2, A_2 \setminus E) \leq \kappa^z(E' \cap A_2, A_2 \setminus E')$
3. $\kappa^z(E \cap A_1, A_1 \setminus E) \leq \kappa^z(E' \cap A_2, A_2 \setminus E')$
4. $\kappa^z(E \cap A_2, A_2 \setminus E) \leq \kappa^z(E' \cap A_1, A_1 \setminus E')$

Case 1 and 2 are clear via definition. For case 3 we know that $\kappa^z(E \cap A_2, A_2 \setminus E) \leq \kappa^z(E \cap A_1, A_1 \setminus E)$ and therefore also $\kappa^z(E \cap A_2, A_2 \setminus E) \leq \kappa^z(E' \cap A_2, A_2 \setminus E')$. Hence, weak decomposition is satisfied. Case 4 can be proven similar to case 3.

The final properties we want to recall are the reinstatement ones, which state that if an argument is defended and does not add conflicts into a set, then the addition of this argument into a set should not lower the plausibility, respectively should raise the plausibility of the set.

**Theorem 6.** $\preceq_{AF}^z$ satisfies weak reinstatement. Where $\tau$ satisfies weak reinstatement iff $a \in F_{AF}(E)$, $a \notin E$ and $a \notin (E^- \cup E^+) \Rightarrow E \cup \{a\} \preceq_{AF}^z E$.

**Proof.** Let $AF = (A, R)$ be an AF and $E \subseteq A$. Assume $a \notin E$ and $a \notin (E^- \cup E^+)$. We have to show that $\kappa^z(E \cup \{a\}, A \setminus E \cup \{a\}) \leq \kappa^z(E, A \setminus E)$. We know that $E \cup \{a\}$ violates the same attacks as $E$, because $E$ and $\{a\}$ are not in conflict with each other. This means that $\kappa^z(E \cup \{a\}, A \setminus E \cup \{a\})$ can not be greater than $\kappa^z(E, A \setminus E)$.

For strong reinstatement i.e., adding an argument into an AF, which is defended by a set and does not create more conflicts, should raise the plausibility, we can look at Example 6. We see, that $\{c\}$ is equally ranked to $\{a, c\}$ despite it holds that $a \in F_{AF}(\{c\})$, $a \notin \{c\}$ and $a \notin (\{c\}^- \cup \{c\}^+)$. So strong reinstatement is violated.

Even though a number of properties are violated by $\preceq_{AF}^z$ this does no lower the impact of this semantics, since $\preceq_{AF}^z$ focuses on a global view. The semantics identifies important attacks in the AF and ensures, that these attacks are satisfied. So the number of not satisfies attacks is not as influential for a ranking of sets, than the impact of attacks. Another difference of this semantics to the semantics of Skiba et al. [2] is the fact, that the number of conflicts a set contains in not important just the fact, that the set is not conflict-free is significant.

6. Discussion

In this work, we continue the research of investigating the relationship of conditional logics and abstract argumentation, by using concepts for conditional logics to reason in abstract argumentation. In particular, we defined a formalism of OCFs to rank sets of arguments. It turns out that these preorder are in line with current work about extension-ranking semantics and produce a ranking for the powerset of arguments for an argumentation framework.

One use of conditional logics is belief change. Where preorder are used to update beliefs with information inconsistent with them. There are a number of different works investigating belief change involving preorder over extensions of an argument-
tation framework [12,13,14]. However, all these works tackle a different problem. To summarise, given an AF and an extension semantics σ, the AF will be changed using a preorder to satisfy new information. This paper talks about using OCFs to reason over sets of argument, while not changing AFs. Weydert [9] investigates a different idea to define extension rankings using conditionals, his definitions could be used to define an extension-ranking semantics similar to Section 5. However, his semantics cannot differentiate conflicting sets. All conflicting sets have the same rank of infinity. A full investigation of the properties of the resulting extension-ranking semantics will be done in future work. A noteworthy mention is that System Z and rational closure by Lehmann and Magidor [15] use the same construction. So our work also allows us to draw connections between argumentation and non-monotonic inference. Additionally OCFs with natural numbers and an infinity level are really close to possibilistic logics [16].

As there are more possible OCFs satisfying our proposed principles, we can define more extension-rankings semantics, like for example an extension-ranking semantics based on c-representations [17].

Acknowledgements. The research reported here was supported by the Deutsche Forschungsgemeinschaft under grant 423456621.

References