Probabilities on Extensions in Abstract Argumentation

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Abstract. Combining computational models of argumentation with probability theory has recently gained increasing attention, in particular with respect to abstract argumentation frameworks. Approaches following this idea can be categorised into the constellations and the epistemic approach. While the former considers probability functions on the subgraphs of abstract argumentation frameworks, the latter uses probability theory to represent degrees of belief in arguments, given a fixed framework. In this paper, we investigate the case where probability functions are given on the extensions of abstract argumentation frameworks. This generalises classical semantics in a straightforward fashion and we show that our approach also complies with many postulates for epistemic probabilistic argumentation.

1 Introduction

Computational models of argumentation are non-monotonic reasoning formalisms that focus on the role of arguments, i. e., defeasible reasons supporting a certain claim, and their relationships. In this context, the well-known formalism of abstract argumentation frameworks [12] abstracts from the inner structure of arguments and only models conflict between them, thus representing argumentation scenarios as directed graphs where arguments are vertices and an attack of one argument on another is modelled by a directed edge. Still, this approach is quite expressive, subsumes many other approaches to non-monotonic reasoning, and provides an active research field. Many research topics have been spawned around these frameworks including, among others, semantical issues [3], extensions on support [11], algorithms [9], and systems [31].

In their original form, abstract argumentation frameworks are a qualitative approach to non-monotonic reasoning as their semantics is set-based (it amounts to identifying sets of collectively acceptable arguments, called *extensions*) and inferences consist of statements regarding the acceptance status of arguments, which can be binary (an argument is simply "accepted" or "rejected") or three-valued (where a third option "undecided" is also possible). In recent years, many approaches have been developed that incorporate some quantitative aspects into abstract argumentation frameworks. These can be categorised into two families. In the first family, the syntactic representation of argumentation frameworks is extended with quantities, in order to incorporate more information explicitly. For example, in [26] arguments and attacks can be annotated with probabilities that model user-supplied information about the likelihood that these objects actually appear in the argumentation framework. This approach is also called the *constellations approach* to probabilistic argumentation [20]. The main aim of these works is then to generalise classical semantics and other notions to the extended approach. See also [13, 34] for some other examples from this family based on weights and fuzzy logic, respectively. The other family is about bringing quantities into the semantics of vanilla argumentation frameworks themselves. Here, the syntactic representation is not extended and the aim is to derive quantitative information which is implicit in the topology of the graph. Concrete approaches within this family are, e. g., numerical ranking functions [1, 19, 7, 28] and the equational approach [17, 18]. The *epistemic approach* to probabilistic argumentation [29, 5, 22, 23] considers the use of probability functions to capture the degrees of belief of an agent in (sets of) arguments (see [22] for a discussion). In this sense the epistemic approach shares some properties with both the families introduced above: on the one hand, the probability values are user-supplied, since they represent the belief of some agent, on the other hand, they can be put in relationship with the semantics of vanilla argumentation frameworks, since it is reasonable to assume that the beliefs of an agent take into account (and/or are constrained by) the topology of the graph.

In this paper, we contribute to the research trend on probabilistic argumentation by considering a further option, which consists in adding a probabilistic layer on top of classical semantics of abstract argumentation frameworks, i. e., we consider probability functions on extensions. This investigation is motivated by the fact that given an argumentation framework, capturing the attacks existing between arguments, each extension prescribed by an argumentation semantics can be regarded as an alternative answer to the question: “which arguments are able to survive the conflict together?”. Thus the set of extensions can be regarded as a set of alternative reasonable options, each satisfying the “survival criterion” encoded by the argumentation semantics, which however does not provide any indication on which extension to select, in case the agents needs to finally choose one of them. This is required in particular in the case of practical reasoning where arguments concern reasons about what to do and alternative extensions may be put in correspondence with different available courses of action. In this context probabilities on extensions may encode additional information, external to the argumentation process, about which option is more likely to be selected by an agent. For instance suppose that in the context of some reasoning activity involving a health problem, two extensions emerge as reasonable, say one corresponding to undergoing surgery and the other to assuming a drug for a long time. The final choice is uncertain and is in the hands of the patient, whose (possibly non-rational) attitude towards the two options can be modeled by a probability assignment on the two extensions, e. g. you may assign a higher probability to the second extension if you know that the patient is particularly worried about the scars caused by surgery. These probability values could be acquired for instance using an approach to probabilistic user modeling, as proposed in [21].

Besides modelling the attitudes of a single agent, probabilities on extensions may be used to model collective attitudes too. Consider the case where two or more politicians argue about their government programmes and assume that their different positions are acceptable from an argumentative point of view. Then a probability assignment on the extensions corresponding to the positions of the candidates may reflect the outcomes of an opinion poll among the voters (note that the use of votes in the context of argumentation frameworks to support an initial numerical assessment, though not of probabilistic nature, has been considered in [25, 15]).

Probability assignments on extensions provide then the basis for further inferential activities, for instance an argument can in general be included in different extensions and it is interesting to consider the probability that a specific argument (or sets of arguments) is selected. In the political example, different candidates, say all candidates, may share the argument that "we should cut taxes since this will promote economical growth", then the probability that this argument is accepted and that tax cuts are in the next government programme is 1, independently of the individual probabilities assigned to the various extensions/candidates (provided that you trust that politicians keep faith with their promises).

Altogether, the general idea is to provide a contribution to the investigation of integrated uncertain reasoning models encompassing both qualitative (in our case, based on abstract argumentation) and quantitative (in our case, probabilistic) evaluation aspects.

To provide a formal basis to this kind of modelling and reasoning activities, in this paper we investigate probability functions on extensions, and in particular,

1. we introduce our approach to probability functions over extensions and we draw some relationships with the maximum entropy principle and with imprecise probabilities (Section 3);
2. we investigate the properties of this extension, in particular wrt. rationality postulates usually considered for the epistemic approach (Section 4);
3. we investigate some computational issues of the approach (Section 5).

Necessary preliminaries are introduced in Section 2 and we conclude with a summary in Section 6.

2 Preliminaries

Abstract argumentation frameworks [12] take a very simple view on argumentation as they abstract away any detail about the internal structure of an argument, its origin and nature and so on. Abstract argumentation frameworks only capture the conflicts between arguments by means of a binary attack relation.

Definition 1. *An* abstract argumentation framework AF *is a tuple* $\mathsf{AF} = (\mathsf{Arg}, \rightarrow)$ *where* Arg *is a set of arguments and* $\rightarrow$ *is a relation* $\rightarrow \subseteq \mathsf{Arg} \times \mathsf{Arg}$.

For the sake of simplicity, in this paper we assume that the set Arg is finite. For two arguments $\mathcal{A}, \mathcal{B} \in \mathsf{Arg}$ the relation $\mathcal{A} \rightarrow \mathcal{B}$ means that argument $\mathcal{A}$ attacks argument $\mathcal{B}$. We abbreviate $\mathsf{Att}_{\mathsf{AF}}(\mathcal{A}) = \{\mathcal{B} \mid \mathcal{B} \rightarrow \mathcal{A}\}$. Abstract argumentation frameworks can be concisely represented by directed graphs, where arguments are represented as nodes and edges model the attack relation.

Example 1. Consider the abstract argumentation framework $\mathsf{AF}_1 = (\mathsf{Arg}_1, \rightarrow_1)$ depicted in Figure 1. Here it is $\mathsf{Arg}_1 = \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$ and $\rightarrow_1 = \{(\mathcal{A}_1, \mathcal{A}_2), (\mathcal{A}_2, \mathcal{A}_1), (\mathcal{A}_2, \mathcal{A}_3), (\mathcal{A}_3, \mathcal{A}_4), (\mathcal{A}_4, \mathcal{A}_5), (\mathcal{A}_5, \mathcal{A}_4), (\mathcal{A}_3, \mathcal{A}_5)\}$.

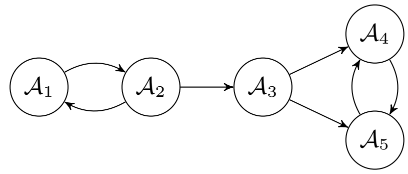

Fig. 1. The argumentation framework AF_1 from Example 1

An argumentation semantics is a formal criterion to determine the conflict outcomes. Two main approaches to semantics definition are available in the literature, namely the extension-based approach [12] and the labeling-based approach [35]. In this paper we focus on the extension-based approach, the reader is referred to [3] for a review and an analysis of the correspondence between the two approaches. An *extension* E of an argumentation framework $\mathsf{AF} = (\mathsf{Arg}, \rightarrow)$ is a set of arguments $E \subseteq \mathsf{Arg}$ that corresponds to a coherent and tenable view in the argumentation process underlying AF. Intuively an extension is a set of arguments which are "collectively acceptable" or "can survive the conflict together".

In the literature [12, 8, 3] a wide variety of different types of semantics has been proposed. The definition of a semantics typically builds on some basic properties that an extension should satisfy: arguably, conflict-freeness and admissibility are among the most important extension properties.

Definition 2. *An extension $E \subseteq \mathsf{Arg}$ is* conflict-free *if for all $\mathcal{A}, \mathcal{B} \in E$ it is not the case that $\mathcal{A} \rightarrow \mathcal{B}$. An extension $E \subseteq \mathsf{Arg}$* defends *an argument $\mathcal{A} \in \mathsf{Arg}$ if for all $\mathcal{C} \in \mathsf{Arg}$, if $\mathcal{C} \rightarrow \mathcal{A}$ then there is $\mathcal{B} \in E$ with $\mathcal{B} \rightarrow \mathcal{C}$. An extension $E \subseteq \mathsf{Arg}$ is admissible if it is conflict-free and defends all its elements.*

We abbreviate by *cf*(AF) the set of conflict-free extensions, by *mcf*(AF) the maximal (wrt. set inclusion) conflict-free extensions, and by *adm*(AF) the set of admissible extensions. Dung's traditional semantics are defined by imposing further constraints.

Definition 3. *Let $\mathsf{AF} = (\mathsf{Arg}, \rightarrow)$ be an abstract argumentation framework and E an admissible extension.*

- *E is* complete *if for all $\mathcal{A} \in \mathsf{Arg}$, if E defends $\mathcal{A}$ then $\mathcal{A} \in E$.*
- *E is* grounded *if and only if E is minimal among complete extensions.*
- *E is* preferred *if and only if E is maximal among complete extensions.*
- *E is* stable *if and only if E is complete and attacks all other arguments.*

All statements on minimality/maximality are meant to be with respect to set inclusion.

We denote by *comp*(AF), *ground*(AF), *pref*(AF), and *st*(AF) the sets of complete, grounded, preferred, and stable extensions of AF, respectively. Note that a grounded extension is uniquely determined and always exists [12], so we also abbreviate by $GE(\mathsf{AF})$ the unique grounded extension of AF, i. e., *ground*(AF) $= \{GE(\mathsf{AF})\}$. Furthermore, we have the following relationships, cf. [3].

Proposition 1. *Let* $\mathsf{AF} = (\mathsf{Arg}, \rightarrow)$ *be an abstract argumentation framework. Then*

1. $st(\mathsf{AF}) \subseteq mcf(\mathsf{AF}) \subseteq cf(\mathsf{AF})$,
2. $st(\mathsf{AF}) \subseteq pref(\mathsf{AF}) \subseteq comp(\mathsf{AF}) \subseteq adm(\mathsf{AF}) \subseteq cf(\mathsf{AF})$, *and*
3. $ground(\mathsf{AF}) \subseteq comp(\mathsf{AF})$.

Besides the above mentioned four traditional semantics, a variety of further proposals have been considered in the literature such as *CF2 semantics* [2], which is not based on the admissibility property. However, in this paper we focus on complete, grounded, preferred, and stable semantics.

Example 2. We continue Example 1. There, the sets $E_1, \ldots, E_6$ given via

$$E_1 = \emptyset \qquad E_2 = \{\mathcal{A}_1\} \qquad E_3 = \{\mathcal{A}_2\}$$
$$E_4 = \{\mathcal{A}_1, \mathcal{A}_3\} \qquad E_5 = \{\mathcal{A}_2, \mathcal{A}_4\} \qquad E_6 = \{\mathcal{A}_2, \mathcal{A}_5\}$$

are admissible. Furthermore, $E_1, E_3, \ldots, E_6$ are complete, E_1 is grounded, and E_4, E_5, E_6 are both preferred and stable.

As shown by the above example, in general argumentation semantics are *multi-extension* or *multiple-status* i. e. they may prescribe more than one extension for a given argumentation framework. When a semantics prescribes exactly one extension for every argumentation framework it is called *single-extension* or *single-status*. Among the semantics considered in this paper, only grounded semantics is single-status.

The possible existence of multiple extensions gives rise to different notions of the justification status of an argument. Given a semantics $\mathcal{S}$, an argument $\mathcal{A}$ is *credulously justified* if there is an $\mathcal{S}$-extension E such that $\mathcal{A} \in E$; $\mathcal{A}$ is *skeptically justified* if for all $\mathcal{S}$-extensions E it holds that $\mathcal{A} \in E$. Note that, unless the set of extensions is empty, being skeptically justified implies being credulously justified and that the two notions coincide for single-extension semantics.

Example 3. We continue Example 2. Here, no argument is skeptically justified wrt. grounded, complete, preferred, and stable semantics. Furthermore, no argument is credulously justified wrt. grounded semantics and all arguments are credulously justified wrt. the other semantics.

3 Probabilities on Extensions

Let $\mathsf{AF} = (\mathsf{Arg}, \rightarrow)$ be fixed. As in the epistemic approach to probabilistic argumentation [29, 5, 22, 23], we consider probability functions on sets of arguments, namely functions $P : 2^{\mathsf{Arg}} \rightarrow [0, 1]$ with

$$\sum_{E \subseteq \mathsf{Arg}} P(E) = 1$$

the idea being that $P(E)$ indicates the probability that the extension E is selected as the final outcome of the semantics evaluation of AF. We denote as $\mathcal{P}_{\mathsf{AF}}$ the set of all such probability functions. For $P_1, P_2 \in \mathcal{P}_{\mathsf{AF}}$ we define $P_1 = P_2$ iff $P_1(E) = P_2(E)$ for all $E \subseteq \mathsf{Arg}$.

Central to our approach is the following definition

	$E_1 = \emptyset$	$E_2 = \{\mathcal{A}_1\}$	$E_3 = \{\mathcal{A}_2\}$	$E_4 = \{\mathcal{A}_1, \mathcal{A}_3\}$	$E_5 = \{\mathcal{A}_2, \mathcal{A}_4\}$	$E_6 = \{\mathcal{A}_2, \mathcal{A}_5\}$
P_1	0.2	0.1	0.3	0.2	0.1	0.1
P_2	0	0.3	0.2	0.3	0.1	0.1
P_3	0	0.2	0.2	0.2	0.2	0.2
P_4	0	0	0	0.3	0.1	0.6
P_5	0	0	0	1/3	1/3	1/3
P_6	0	0	0	0.5	0.5	0.0
P_7	1	0	0	0	0	0

Table 1. Definition of probability functions from Example 4; $P_i(E) = 0$ for all remaining $E \notin \{E_1, \ldots, E_6\}$ for $i = 1, \ldots, 7$

Definition 4. *We say that $P \in \mathcal{P}_{\mathsf{AF}}$ is* semantically based *on a set $\mathcal{E} \subseteq 2^{\mathsf{Arg}}$, if $P(E) = 0$ for all $E \notin \mathcal{E}$.*

We denote as $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}} \subseteq \mathcal{P}_{\mathsf{AF}}$ the set of all probability functions that are semantically based on $\mathcal{E}$. For example, $\mathcal{P}_{\mathsf{AF}}^{mcf(\mathsf{AF})}$ is the set of all probability functions that are semantically based on the maximal conflict-free subsets of AF. Note that in many cases one can assume that the set $\mathcal{E}$ is known a priori, e.g. the set of extensions prescribed by a given semantics for a given argumentation framework can be computed using one of the available implemented systems for abstract argumentation [10, 32, 30]. In this case one can of course easily ensure that a probability function is semantically based on $\mathcal{E}$ by construction. The issue of studying computational procedures for indirectly enforcing that a probability function is semantically based on a set $\mathcal{E}$ and for transforming an arbitrary probability function into the "closest" one which is semantically based on a given set $\mathcal{E}$ are interesting issues of future work.

Example 4. We continue Example 3 and consider the probability functions $P_1, \ldots, P_7$ defined in Table 1. All these functions are semantically based on the admissible sets of AF_0, i. e., $P_1, \ldots, P_7 \in \mathcal{P}_{\mathsf{AF}_1}^{adm(\mathsf{AF}_1)}$. Furthermore, we have

- $P_4, \ldots, P_7 \in \mathcal{P}_{\mathsf{AF}_1}^{comp(\mathsf{AF}_1)}$,
- $P_4, P_5, P_6 \in \mathcal{P}_{\mathsf{AF}_1}^{st(\mathsf{AF}_1)} = \mathcal{P}_{\mathsf{AF}_1}^{pref(\mathsf{AF}_1)}$, and
- $P_7 \in \mathcal{P}_{\mathsf{AF}_1}^{ground(\mathsf{AF}_1)}$.

A first observation is that we obtain the same hierarchy of the probabilistic versions of semantics as in Proposition 1.

Proposition 2. *If $\mathcal{E} \subseteq \mathcal{E}'$ then $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}} \subseteq \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}'}$. In particular*

1. $\mathcal{P}_{\mathsf{AF}}^{st(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{mcf(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{cf(\mathsf{AF})}$,
2. $\mathcal{P}_{\mathsf{AF}}^{st(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{pref(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{comp(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{adm(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{cf(\mathsf{AF})}$, *and*
3. $\mathcal{P}_{\mathsf{AF}}^{ground(\mathsf{AF})} \subseteq \mathcal{P}_{\mathsf{AF}}^{comp(\mathsf{AF})}$.

Proof. *This follows directly from Definition 4 and Proposition 1.* □

Furthermore, as in the classical case we have that probabilistic reasoning wrt. grounded semantics is uniquely defined.

Proposition 3. $|\mathcal{P}_{\mathsf{AF}}^{\mathit{ground}(\mathsf{AF})}| = 1$.

Proof. As AF *has a unique grounded extension E, any P semantically based on grounded semantics must have $P(E) = 1$ and $P(E') = 0$ for all other sets E'. Therefore, P is uniquely determined.* □

Given a probability function $P \in \mathcal{P}_{\mathsf{AF}}$ representing uncertainty about which extension is selected, an agent may be focused on a single argument or, more generally on a set of arguments, and be interested in the probability that this argument or sets of arguments is included in the selected extension E. In other words the probability P can be extended to the events of the kind $(F \subseteq E)$ where F is a generic set of arguments and E is the selected extension. For a set of arguments F, this extended probability will be denoted as $P^{\subseteq}(E)$ and is derived from P as follows

$$P^{\subseteq}(F) = \sum_{E \in 2^{\mathsf{Arg}}, F \subseteq E} P(E) \tag{1}$$

For individual arguments $\mathcal{A} \in \mathsf{Arg}$ we introduce a special notation

$$P^{\in}(\mathcal{A}) \triangleq P^{\subseteq}(\{\mathcal{A}\}) = \sum_{E \in 2^{\mathsf{Arg}}, \mathcal{A} \in E} P(E) \tag{2}$$

Example 5. Continuing Example 4, we have, e. g.

$$P_2^{\in}(\mathcal{A}_2) = P_2(E_3) + P_2(E_5) + P_2(E_6) = 0.4$$
$$P_4^{\in}(\mathcal{A}_5) = P_4(E_6) = 0.6$$

The following propositions report some basic observations.

Proposition 4. *For $P \in \mathcal{P}_{\mathsf{AF}}^{\mathit{cf}(\mathsf{AF})}$, $P^{\in}(\mathcal{A}) = 0$ for all self-attacking arguments $\mathcal{A}$.*

Proof. If $\mathcal{A}$ is self-attacking then $\mathcal{A}$ is not member of any conflict-free set E of AF*. Therefore $P^{\in}(\mathcal{A}) = \sum_{\mathcal{A} \in E \in \mathit{cf}(\mathsf{AF})} P(E) = 0$.* □

Proposition 5. *For $P \in \mathcal{P}_{\mathsf{AF}}^{\mathit{comp}(\mathsf{AF})}$, $P^{\subseteq}(GE(\mathsf{AF})) = 1$ and $P^{\in}(\mathcal{A}) = 1$ for every argument $\mathcal{A} \in GE(\mathsf{AF})$.*

Proof. The statement follows from the fact that the grounded extension of AF *is included in every complete extension of* AF*.* □

While some basic results, as shown above, hold for every probability function P, provided that P is semantically based on a given set of extensions, more specific properties of the beliefs of an agent may depend on the actual probability function P adopted by the agent within $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$. In case an agent has no information or criteria to adopt a specific P, the well-known *maximum entropy principle* [27, 24] states that the uniform probability assignment is adopted. In our case, the assignment of uniform nonzero probability values is restricted to the prescribed set of extensions.

Definition 5. *Let $P \in \mathcal{P}_{\mathsf{AF}}$. We say that P is* semantically uniform *on $\mathcal{E} \subseteq 2^{\mathsf{Arg}}$, if $P \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ and for all $E, E' \in \mathcal{E}$ we have $P(E) = P(E')$.*

Of course semantical uniform probability functions are uniquely determined, given AF and $\mathcal{E}$ and the value of $P^{\in}(\mathcal{A})$ for each argument $\mathcal{A}$ is easily characterised.

Proposition 6. *Let $\mathcal{E} \subseteq 2^{\mathsf{Arg}}$.*

1. *If $P, P' \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ are semantically uniform on $\mathcal{E}$, then $P = P'$, i.e. $\forall E \in \mathcal{E}\ P(E) = P'(E)$.*
2. *If $P \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ is semantically uniform on $\mathcal{E}$, then for all $\mathcal{A} \in \mathsf{Arg}$*

$$P^{\in}(\mathcal{A}) = \frac{|\{E \in \mathcal{E} \mid \mathcal{A} \in E\}|}{|\mathcal{E}|}$$

Proof. *This follows directly from Definition 5.* $\square$

Also we are interested to characterise the case where the set of possible extensions is restricted (e. g. from admissible extensions to complete extensions) while still applying the maximum entropy principle.

Definition 6. *$P \in \mathcal{P}_{\mathsf{AF}}$ is a* semantical uniform restriction *of $P' \in \mathcal{P}_{\mathsf{AF}}$, if P is semantically uniform on $\mathcal{E}$, P' is semantically uniform on $\mathcal{E}'$, and $\mathcal{E} \subseteq \mathcal{E}'$.*

Example 6. We continue Example 4. While both P_2 and P_3 are semantically based on $\mathcal{E} = \{E_2, \ldots, E_6\}$, only P_3 is semantically uniform wrt. $\mathcal{E}$. Furthermore, P_4, P_5, P_6 are semantically based on the stable/preferred extensions and P_5 is also semantically uniform on those. P_5 is also a semantical uniform restriction of P_3 and P_6 is a semantical uniform restriction of P_5.

The maximum entropy principle offers a simple criterion to select one representative element in the (usually uncountably large) set of probability functions that are semantically based on some set of extensions. By construction, the information content of this representative element is rather weak: in particular, as to individual arguments, it boils down to counting how often an argument appears in extensions, cf. item 2 of Proposition 6.

In general, given a set of probability functions, their *lower envelope* [33] can be regarded as another synthetic representative of the set itself.

Definition 7. *Given a set of probability functions $\mathcal{P}$ on a set $\mathcal{E}$ the lower envelope $\underline{P}$ of $\mathcal{P}$ is defined for each $E \in \mathcal{E}$ as $\underline{P}(E) = \inf_{P \in \mathcal{P}} P(E)$.*

The lower envelope of a set of probabilities has interesting formal properties since it belongs to the family of imprecise probabilities and in particular is a *coherent lower probability* [33] (see Theorem 1 below). In words, $\underline{P}(E)$ identifies the minimum degree of belief in E given the set $\mathcal{P}$. The function $\underline{P}$ can therefore be regarded as a sort of cautious representation of the information content of $\mathcal{P}$. Specialising this notion to our context we get the following definition.

Definition 8. *Given a set of probability functions $\mathcal{P} \subseteq \mathcal{P}_{\mathsf{AF}}$ we define*[1]

- $\underline{P}(E) = \inf_{P \in \mathcal{P}} P(E)$ *for every* $E \in 2^{\mathsf{Arg}}$
- $\underline{P}^{\subseteq}(E) = \inf_{P \in \mathcal{P}} P^{\subseteq}(E)$ *for every* $E \in 2^{\mathsf{Arg}}$
- $\underline{P}^{\in}(\mathcal{A}) = \inf_{P \in \mathcal{P}} P^{\in}(\mathcal{A})$ *for every* $\mathcal{A} \in \mathsf{Arg}$

It is worth noting that each coherent lower probability $\underline{P}$ function has a conjugate upper probability $\overline{P}$ which for each E is defined by the following conjugacy relation

$$\overline{P}(E) = 1 - \underline{P}(\neg E) \tag{3}$$

Thus for instance the upper probability that a given extension E is selected is equal to 1 minus the lower probability that E is not selected. Given the set of probability functions $\mathcal{P}$ of which $\underline{P}$ is the lower envelope, $\overline{P}$ can be equivalently characterized as the upper envelope of $\mathcal{P}$, replacing $\inf$ with $\sup$ and making other obvious adjustments in Definitions 7 and 8. In this sense, dually with respect to $\underline{P}$, the function $\overline{P}$ can be regarded as a sort of optimistic representation of the information content of $\mathcal{P}$.

In general, for an event E, the interval $[\underline{P}(E), \overline{P}(E)]$ gives an account of the distance between a cautious and an optimistic reading of the set $\mathcal{P}$ with respect to E. In particular if $\underline{P}(E) = \overline{P}(E)$, the set $\mathcal{P}$ provides a precise information about the probability of E, while at the other extreme, if $\underline{P}(E) = 0$ and $\overline{P}(E) = 1$, the set $\mathcal{P}$ provides no information at all about the probability of E.

The reader is referred to [33] for an extensive treatment of these concepts. In particular in [33] the values $\underline{P}(E)$ and $\overline{P}(E)$ were given a behavioral interpretation in an idealized betting scheme on E.

To make this notion clearer, we recall that this interpretation is rooted in De Finetti's *subjective probability theory* [16], of which the theory of imprecise probabilities introduced in [33] is a generalisation.

In De Finetti's approach a (precise) probability assessment is a function $P : \mathcal{E} \to \mathbb{R}$, where $\mathcal{E}$ is an *arbitrary* (finite or infinite) set of events and $\mathbb{R}$ is the set of real numbers. For each event $E \in \mathcal{E}$, $P(E)$ is the "fair" price of a (unitary) bet on E, i.e. $P(E)$ is the amount of money that an agent is ready to pay to an opponent in order to receive the sum of 1 if E turns out to be **`true`** and 0 otherwise, and, indifferently, the sum that the agent is ready to receive from an opponent as a payment for the commitment to pay the sum of 1 if E turns out to be **`true`** and 0 otherwise. More formally, $P(E)$ is the price, according to the agent, of the *indicator* of E, denoted as $I(E)$, namely the random number which takes value 1 if E is **`true`**, and value 0 if E is **`false`**. It is assumed that the agent is indifferently ready to buy or sell $I(E)$ at price $P(E)$. In the case of buying, the random gain of the agent is $I(E) - P(E)$, while it is $P(E) - I(E)$ in the case of selling. A not necessarily unitary bet is characterized by a real coefficient (or stake) $s \in \mathbb{R}$, so that the gain of the agent is given by $s(I(E) - P(E))$. A positive (negative) value of s corresponds to a buying (selling) choice by the agent.

According to the betting interpretation, a probability assessment has to satisfy some conditions ensuring that the bet makes sense for both participants. In particular, de Finetti has established a property of coherence, called *dF-coherence* in the sequel.

[1] Note that the definitional relation for $P^{\subseteq}(E)$ in (1) does not carry over to $\underline{P}^{\subseteq}(E)$, i.e. in general it does not hold that $\underline{P}^{\subseteq}(E) = \sum_{E' \in 2^{\mathsf{Arg}}, E \subseteq E'} \underline{P}(E')$. An analogous consideration applies to $\underline{P}^{\in}(\mathcal{A})$.

Definition 9. *Given an arbitrary set of events $\mathcal{E}$, $P : \mathcal{E} \to \mathbb{R}$ is a dF-coherent probability if and only if $\forall n \in \mathbb{N}^+$, $\forall s_1, \ldots, s_n \in \mathbb{R}$, $\forall E_1, \ldots E_n \in \mathcal{E}$, it holds that*

$$max \left[\sum_{i=1}^{n} s_i(I(E_i) - P(E_i)) \right] \geq 0 \tag{4}$$

where $\mathbb{N}^+$ is the set of positive integer numbers.

Intuitively dF-coherence states that for any finite combination of bets, the maximum value of the random gain of the agent is non-negative, hence the agent avoids a sure loss. It is well-known that dF-coherence implies several fundamental properties[2] of probability assessments, including in particular the fact that $0 \leq P(E) \leq 1$ for every event E and the following *self-conjugacy* relation:

$$P(E) = 1 - P(\neg E). \tag{5}$$

Considering the same betting context, imprecise probabilities [33] can be introduced by lifting the assumption that the agent has a precise price estimation, used indifferently for buying or selling event indicators. Rather (as typical in real markets) the agent considers, for each event E, two different prices, one for buying and one for selling $I(E)$, denoted respectively as $\underline{P}(E)$ and $\overline{P}(E)$. Clearly, $\underline{P}(E) \leq \overline{P}(E)$. Moreover, the agent is of course ready to buy also at any price lesser than $\underline{P}(E)$, which hence represents the *supremum buying price* for $I(E)$. Similarly, $\overline{P}(E)$ is the *infimum selling price* for $I(E)$. Given that, for any event E, $I(\neg E) = 1 - I(E)$, it turns out that buying an event is equivalent to selling its complement and vice versa. Hence, in the context of imprecise probabilities, the following *conjugacy relation* replaces condition (5):

$$\underline{P}(E) = 1 - \overline{P}(\neg E) \tag{6}$$

In virtue of the conjugacy relation, one can focus on lower or upper probabilities only.

Definition 10 provides the notion of coherence for lower probabilities [33].

Definition 10. *Given an arbitrary set of events $\mathcal{E}$, $\underline{P} : \mathcal{E} \to \mathbb{R}$ is a coherent lower probability if and only if $\forall n \in \mathbb{N} = \mathbb{N}^+ \cup \{0\}$, and for all real and* non-negative *$s_0, \ldots, s_n$, $\forall E_0, \ldots E_n \in \mathcal{E}$, it holds that*

$$max \left[\left[\sum_{i=1}^{n} s_i(I(E_i) - \underline{P}(E_i)) \right] - s_0(I(E_0) - \underline{P}(E_0)) \right] \geq 0 \tag{7}$$

The coherence condition requires that the maximum of the gain of the agent is non negative for every (including the empty) combination of buying bets with at most one selling bet of a single (arbitrarily selected) event E_0. In a sense Definition 10 allows the agent to use its supremum buying price for any buying transaction but also forces the agent to use the same price for (at most one) selling transaction. Intuitively, this

[2] For finite sets of events, essentially all properties of "traditional" probability functions are recovered.

ensures that the assessment $\underline{P}$ by the agent is not too unfair. Actually, it has been shown in [33] that, in the context of imprecise probabilities, the property of coherence is a strengthening of the property of avoiding sure loss.

As already mentioned, the lower envelope theorem, one of the main results of the theory of imprecise probabilities developed in [33], provides a nice characterization of coherent lower probabilities by relating them to sets of precise probabilities.

Theorem 1 ([33]). *Given a set $\mathcal{E}$, $\underline{P}$ is a coherent lower probability on $\mathcal{E}$ if and only if there is a set $\mathcal{P}$ of (precise) dF-coherent probabilities on $\mathcal{E}$ such that $\underline{P}(E) = \inf_{P \in \mathcal{P}} P(E)$ for every $E \in \mathcal{E}$.*

In words, a lower probability $\underline{P}$ is coherent if and only if it can be obtained as the lower envelope of a set ($\mathcal{P}$) of dF-coherent precise probabilities (P). This result provides both a constructive procedure for coherent lower probabilities and a motivation for their existence: when a set of different probability assessments is given, coherent lower probabilities arise by aggregating them in the least committed way.

Example 7. With reference to Table 1, let $\mathcal{P} = \{P_1, \ldots, P_6\}$, $\underline{P}$ be its lower envelope and $\overline{P}$ its conjugate upper envelope. We have $\underline{P}(E_1) = \underline{P}(E_2) = \underline{P}(E_3) = \underline{P}(E_6) = 0$; $\underline{P}(E_4) = 0.2$; $\underline{P}(E_5) = 0.1$ and $\overline{P}(E_1) = 0.2$; $\overline{P}(E_2) = \overline{P}(E_3) = 0.3$; $\overline{P}(E_4) = \overline{P}(E_5) = 0.5$; $\overline{P}(E_6) = 0.6$. Also, for instance, $\underline{P}^{\in}(\mathcal{A}_2) = \inf_{P \in \mathcal{P}}\{P(E_3)+P(E_5)+P(E_6)\} = 0.4$ and dually $\overline{P}^{\in}(\mathcal{A}_2) = \sup_{P \in \mathcal{P}}\{P(E_3) + P(E_5) + P(E_6)\} = 0.7$. We have also $\underline{P}^{\in}(\mathcal{A}_1) = 0.3$; $\overline{P}^{\in}(\mathcal{A}_1) = 0.6$; $\underline{P}^{\in}(\mathcal{A}_3) = 0.2$; $\overline{P}^{\in}(\mathcal{A}_3) = 0.5$; $\underline{P}^{\in}(\mathcal{A}_4) = 0.1$; $\overline{P}^{\in}(\mathcal{A}_4) = 0.5$; $\underline{P}^{\in}(\mathcal{A}_5) = 0$; $\overline{P}^{\in}(\mathcal{A}_5) = 0.6$.

When the set $\mathcal{P}$ coincides with the set $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ of *all* probability functions that are semantically based on $\mathcal{E}$, then for each argument $\mathcal{A}$ the possible values of $\underline{P}(\mathcal{A})$ and $\overline{P}(\mathcal{A})$ are limited, so that the provided information is either extremely precise (both values are either 0 or 1) or completely vague ($\underline{P}(\mathcal{A}) = 0$ and $\overline{P}(\mathcal{A}) = 1$).

Proposition 7. *Given the set of probability functions $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ for some set of extensions $\mathcal{E}$, let $\underline{P}$ be its lower envelope and $\overline{P}$ its conjugate upper envelope. For each argument $\mathcal{A} \in \mathsf{Arg}$ it holds that:*

- $\underline{P}^{\in}(\mathcal{A}) = 1$ *iff* $\forall E \in \mathcal{E}\ \mathcal{A} \in E$; $\underline{P}^{\in}(\mathcal{A}) = 0$ *otherwise;*
- $\overline{P}^{\in}(\mathcal{A}) = 1$ *iff* $\exists E \in \mathcal{E} : \mathcal{A} \in E$; $\overline{P}^{\in}(\mathcal{A}) = 0$ *otherwise.*

Proof. If $\forall E \in \mathcal{E}\ \mathcal{A} \in E$ then $\forall P \in \mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ it holds $P^{\in}(\mathcal{A}) = 1$ from which $\underline{P}^{\in}(\mathcal{A}) = \overline{P}^{\in}(\mathcal{A}) = 1$. Otherwise if $\exists E \in \mathcal{E} :\ \mathcal{A} \notin E$ then the probability function given by $P(E) = 1$ and $P(E') = 0$ for every $E' \neq E$ belongs to $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ from which $P^{\in}(\mathcal{A}) = 0$ and $\underline{P}^{\in}(\mathcal{A}) = 0$. Analogously, if $\exists E \in \mathcal{E} : \mathcal{A} \in E$ the probability function given by $P(E) = 1$ and $P(E') = 0$ for every $E' \neq E$ belongs to $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ from which $P^{\in}(\mathcal{A}) = 1$ and $\overline{P}^{\in}(\mathcal{A}) = 1$. Otherwise $\nexists E \in \mathcal{E} : \mathcal{A} \in E$ and then $\forall P \in \mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ it holds $P^{\in}(\mathcal{A}) = 0$ from which $\overline{P}^{\in}(\mathcal{A}) = 0$.

In general, the lower (or upper) envelope and the upper envelope of a set of precise probabilities are not precise probabilities themselves. However in some special cases some interesting correspondences between lower (or upper) values and precise probability assignments can be obtained. This is in particular the case when considering the set $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ of all probability functions that are semantically based on $\mathcal{E}$: it can be seen that for each argument $\mathcal{A}$ the lower probability value $\underline{P}^{\in}(\mathcal{A})$ induced by the lower envelope of $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ coincides with the precise probability value $P^{\in}(\mathcal{A})$ induced by the precise probability $P \in \mathcal{P}_{\mathsf{AF}}$ which gives probability 1 to the intersection of the elements of $\mathcal{E}$.

Proposition 8. *Given the set of probability functions $\mathcal{P}^{\mathcal{E}}_{\mathsf{AF}}$ for some set of extensions $\mathcal{E}$, let $\underline{P}$ be its lower envelope and let $P \in \mathcal{P}_{\mathsf{AF}}$ be defined as $P(\bigcap_{E\in\mathcal{E}} E) = 1$, $P(E') = 0$ for every $E' \neq \bigcap_{E\in\mathcal{E}} E$. For each argument $\mathcal{A} \in \mathsf{Arg}$ it holds that $\underline{P}^{\in}(\mathcal{A}) = P^{\in}(\mathcal{A})$.*

Proof. By definition, $P^{\in}(\mathcal{A}) = 1$ if $\mathcal{A} \in \bigcap_{E\in\mathcal{E}} E$, $P^{\in}(\mathcal{A}) = 0$ otherwise. From Proposition 7 we have $\underline{P}^{\in}(\mathcal{A}) = 1$ if $\mathcal{A} \in \bigcap_{E\in\mathcal{E}} E$, $\underline{P}^{\in}(\mathcal{A}) = 0$ otherwise, which proves the statement.

A corollary of Proposition 8 concerns the set $\mathcal{P}^{comp(\mathsf{AF})}_{\mathsf{AF}}$ of probabilities semantically based on complete extensions. It follows from the fact that the grounded extension is the least complete extension and coincides with the intersection of all complete extensions and provides a nice counterpart of Proposition 5.

Corollary 1. *Given the set of probability functions $\mathcal{P}^{comp(\mathsf{AF})}_{\mathsf{AF}}$, let $\underline{P}$ be its lower envelope and let P be the unique member of $\mathcal{P}^{ground(\mathsf{AF})}_{\mathsf{AF}}$. For each argument $\mathcal{A} \in \mathsf{Arg}$ it holds that $\underline{P}^{\in}(\mathcal{A}) = P^{\in}(\mathcal{A})$.*

In general, similar considerations could be applied to strict subsets of $\mathcal{P}_{\mathsf{AF}}$ (e. g. satisfying some constraints induced by the beliefs of the considered agent(s)) in order to identify some representative and/or to analyse their information contents. This line of development is left to future work.

4 Comparison to Epistemic Probabilistic Argumentation

In this section we analyze our approach to semantically based probabilities with respect to some general properties considered in the literature for the epistemic approach [29, 5, 22, 23].

First, unattacked arguments play a special role as they are, in a sense, unquestioned. The *Foundation* postulate from [22] requires that the probability of unattacked arguments is 1. In our context this is guaranteed if a probability function is based on a semantic notion at least as strong as completeness.

Proposition 9. *If $P \in \mathcal{P}^{comp(\mathsf{AF})}_{\mathsf{AF}}$ then $P^{\in}(\mathcal{A}) = 1$ for all unattacked arguments $\mathcal{A}$.*

Proof. If $\mathcal{A}$ is not attacked in AF then $\mathcal{A} \in E$ for every complete extension E of AF. Then $P^{\in}(\mathcal{A}) = \sum_{E\in comp(\mathsf{AF})} P(E) = 1$. □

Furthermore, a central postulate in the above-mentioned approaches is *Coherence*, which states that the sum of the probabilities of two conflicting arguments must be at most one. In our context, conflict freeness is enough to guarantee this property.

Proposition 10. *If $P \in \mathcal{P}_{\mathsf{AF}}^{cf(\mathsf{AF})}$ then for every $\mathcal{A}, \mathcal{B} \in \mathsf{Arg}$ with $\mathcal{A} \rightarrow \mathcal{B}$, $P^{\in}(\mathcal{B}) \leq 1 - P^{\in}(\mathcal{A})$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathsf{Arg}$ with $\mathcal{A} \rightarrow \mathcal{B}$. Then for every $E \in cf(\mathsf{AF})$ it cannot be the case that both $\mathcal{A} \in E$ and $\mathcal{B} \in E$. Therefore

$$\begin{aligned} P^{\in}(\mathcal{A}) + P^{\in}(\mathcal{B}) &= \sum_{\mathcal{A} \in E \subseteq \mathsf{Arg}} P(E) + \sum_{\mathcal{B} \in E \subseteq \mathsf{Arg}} P(E) \\ &= \sum_{\mathcal{A} \in E \in cf(\mathsf{AF})} P(E) + \sum_{\mathcal{B} \in E \in cf(\mathsf{AF})} P(E) \\ &\leq \sum_{E \in cf(\mathsf{AF})} P(E) = 1 \end{aligned}$$

□

The *Rationality* postulate [20] states that if an argument has a probability greater than 0.5 then any conflicting argument should have a probability lesser than 0.5. Since this property is implied by *Coherence*, we directly obtain the satisfaction of the Rationality postulate too.

Corollary 2. *If $P \in \mathcal{P}_{\mathsf{AF}}^{cf(\mathsf{AF})}$ then for every $\mathcal{A}, \mathcal{B} \in \mathsf{Arg}$ with $\mathcal{A} \rightarrow \mathcal{B}$, if $P^{\in}(\mathcal{A}) > 0.5$ then $P^{\in}(\mathcal{B}) \leq 0.5$.*

The postulate *Optimism* has been used [29] to establish a certain correspondence to traditional semantics. It states that the sum of the probability of an argument and the probabilities of its attackers should be at least 1. In our context this holds under stable semantics.

Proposition 11. *If $P \in \mathcal{P}_{\mathsf{AF}}^{st(\mathsf{AF})}$ then for every $E \in st(\mathsf{AF})$, $\mathcal{A} \in E$, $P^{\in}(\mathcal{A}) \geq 1 - \sum_{\mathcal{B} \rightarrow \mathcal{A}} P^{\in}(\mathcal{B})$.*

Proof. We have that $P^{\in}(\mathcal{A}) = 1 - \sum_{\mathcal{A} \notin E} P(E)$. By definition every stable extension S attacks all arguments not included in S. Then in particular every stable extension not including $\mathcal{A}$ includes an attacker of $\mathcal{A}$ from which it follows that $\sum_{\mathcal{B} \rightarrow \mathcal{A}} P^{\in}(\mathcal{B}) \geq \sum_{\mathcal{A} \notin E} P(E)$ from which $P^{\in}(\mathcal{A}) \geq 1 - \sum_{\mathcal{B} \rightarrow \mathcal{A}} P^{\in}(\mathcal{B})$.

Moreover two extreme cases have been considered in [22], namely *maximal* (respectively, *minimal*) epistemic probabilities where the probability of every argument is 1 (respectively 0). In our context they can be put in direct correspondence with special topological cases. Assuming probabilities which are semantically based on conflict-free sets, a maximal probability can be obtained only for argumentation frameworks with an empty attack relation.

Proposition 12. *If $P \in \mathcal{P}_{\mathsf{AF}}^{cf(\mathsf{AF})}$ then $P^{\in}(\mathcal{A}) = 1$ for every argument $\mathcal{A} \in \mathsf{Arg}$ only if $\rightarrow = \emptyset$.*

Proof. From the fact that $P^{\in}(\mathcal{A}) = 1$ for every argument $\mathcal{A} \in \mathsf{Arg}$ it follows that it must be the case that $P(\mathsf{Arg}) = 1$ and $P(E) = 0$ for every E such that $E \subsetneq \mathsf{Arg}$. For such a probability P to belong to $\mathcal{P}_{\mathsf{AF}}^{cf(\mathsf{AF})}$ it must be the case that Arg is conflict-free, i.e. $\rightarrow = \emptyset$.

By the way when $\rightarrow = \emptyset$, the whole set of arguments Arg is the unique extension prescribed by all semantics considered in this paper but the conflict-free and the admissible semantics. Thus the maximal probability is also the unique probability compatible with those semantics when no attacks are present.

Conversely, it is clear thar a minimal probability is achieved only when the empty set has probability 1.

Proposition 13. *For $P \in \mathcal{P}_{\mathsf{AF}}$ then $P^{\in}(\mathcal{A}) = 0$ for every argument $\mathcal{A} \in \mathsf{Arg}$ if and only if $P(\emptyset) = 1$ and $P(E) = 0$ for every E such that $\emptyset \subsetneq E \subseteq \mathsf{Arg}$.*

Then, the minimal probability can be semantically based only if the empty set belongs to the set of extensions. In particular, the following proposition is directly derived from basic properties of the grounded and complete semantics (and is related with Proposition 5).

Proposition 14. *Let $P_{\emptyset} \in \mathcal{P}_{\mathsf{AF}}$ be defined as $P_{\emptyset}(\emptyset) = 1$ and $P_{\emptyset}(E) = 0$ for every E such that $\emptyset \neq E \subseteq \mathsf{Arg}$. $P_{\emptyset} \in \mathcal{P}_{\mathsf{AF}}^{comp(\mathsf{AF})}$ iff $GE(\mathsf{AF}) = \emptyset$ iff $\forall \mathcal{A} \in \mathsf{Arg}\ \exists \mathcal{B} \in \mathsf{Arg} : \mathcal{B} \rightarrow \mathcal{A}$.*

Proof. $P_{\emptyset} \in \mathcal{P}_{\mathsf{AF}}^{comp(\mathsf{AF})}$ holds if and only if the empty set is a complete extension, which in turn holds if and only if $GE(\mathsf{AF}) = \emptyset$, given that the grounded extension $GE(\mathsf{AF})$ is the minimal complete extension. By well-known properties of the grounded semantics [12] $GE(\mathsf{AF}) = \emptyset$ holds if and only if every argument has at least an attacker (since every unattacked argument belongs to $GE(\mathsf{AF})$).

5 Computational Issues

We now discuss some computational issues of our approach, in particular, we make some straightforward comments on computational complexity.

Our approach is about probabilistic reasoning [27] with abstract argumentation frameworks. In general, bringing quantities into a qualitative reasoning problem also adds computational complexity. When reasoning with infinite sets such as $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ several properties of this set ensure that this can be done effectively. The next result shows that the set $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$is well-behaved wrt. important properties.

Proposition 15. *For every $\mathcal{E} \subseteq 2^{\mathsf{Arg}}$, $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ is a connected, closed, and convex set.*

Proof. *Let $P_1, P_2 \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$, $\delta \in (0, 1)$, and define the δ-convex combination $P_3 \in \mathcal{P}_{\mathsf{AF}}$ of P_1 and P_2 via*

$$P_3(E) = \delta P_1(E) + (1 - \delta) P_2(E)$$

for all $E \subseteq 2^{\mathsf{Arg}}$. Then for $E' \notin \mathcal{E}$ we have

$$P_3(E') = \delta P_1(E') + (1 - \delta) P_2(E') = 0$$

and therefore $P_3 \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ showing that $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ is convex. Every convex set is also connected.

To show closure, let $P_1, P_2, \ldots$ be a sequence of probability functions in $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ such that $\lim_{i\to\infty} P_i(E)$ exists for all $E \subseteq 2^{\mathsf{Arg}}$ and define $P \in \mathcal{P}_{\mathsf{AF}}$ via

$$P(E) = \lim_{i\to\infty} P_i(E)$$

Note that it is straightforward to see that indeed $P \in \mathcal{P}_{\mathsf{AF}}$. Then for $E' \notin \mathcal{E}$ we have

$$P(E') = \lim_{i\to\infty} P_i(E') = \lim_{i\to\infty} 0 = 0$$

and therefore $P \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ showing that $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ is closed. □

Note that due to the above result pertaining the closure of sets $\mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$, we can substitute "infimum" by "minimum" in Definition 8. Due to connectedness and convexity, minima and maxima can be effectively computed by convex optimisation techniques[3]. We are currently investigating how to exploit this for algorithmic issues.

Regarding computational complexity, the following result immediately follows from well-known complexity results for abstract argumentation, see e. g. [14].

Proposition 16. *Let AF be an abstract argumentation framework and $P \in \mathcal{P}_{\mathsf{AF}}^{\mathcal{E}}$ semantically uniform.*

1. *Deciding whether $P(\mathcal{A}) > 0$ for some $\mathcal{A} \in \mathsf{Arg}$ is*
   - *(a) NP-complete for $\mathcal{E} = comp(\mathsf{AF})$,*
   - *(b) NP-complete for $\mathcal{E} = pref(\mathsf{AF})$,*
   - *(c) NP-complete for $\mathcal{E} = st(\mathsf{AF})$, and*
   - *(d) in P for $\mathcal{E} = ground(\mathsf{AF})$.*
2. *Deciding whether $P(\mathcal{A}) = 1$ for some $\mathcal{A} \in \mathsf{Arg}$ is*
   - *(a) in P for $\mathcal{E} = comp(\mathsf{AF})$,*
   - *(b) Π_2^P-complete for $\mathcal{E} = pref(\mathsf{AF})$,*
   - *(c) coNP-complete for $\mathcal{E} = st(\mathsf{AF})$, and*
   - *(d) in P for $\mathcal{E} = ground(\mathsf{AF})$.*

Proof. Observe that $P(\mathcal{A}) > 0$ is equivalent to asking whether $\mathcal{A}$ is credulously inferred. Correspondingly, $P(\mathcal{A}) = 1$ is equivalent to asking whether $\mathcal{A}$ is skeptically inferred. For the complexity of these problems see e. g. [14]. □

6 Summary

We proposed a novel perspective to combine probability theory with abstract argumentation. In our approach, we combine classical extension-based semantics with quantitative uncertainty by considering probability functions on extensions and analysing some relevant reasoning tasks. We did some preliminary investigation and showed that our

[3] The size of the optimization problem depends of course on the size of the set $\mathcal{E}$ which might be large in some cases. The reader may refer to [4, 6] for studies on the size of the set of extensions prescribed by a given semantics.

proposal faithfully generalises classical semantics and is compatible with some postulates considered in the epistemic approach to probabilistic argumentation. Some relationships with imprecise probability theory were also pointed out and finally, we made some observations regarding computational complexity.

The work reported in this paper is preliminary and a deeper investigation of the proposed formalism and of its potential applications is called for. In particular, the development of algorithmic approaches for using our framework is part of ongoing work. Finally, concerning the issue of where do the probability values come from, we suggest that an interesting direction of investigation is learning or estimating the probabilities of extensions or of arguments from the past choices of an agent or of a community of agents (e. g. an electoral body) in similar decision contexts.

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